AN INTRODUCTION TO THE STUDY OF MAP PROJECTION

JAMES MAINWARING

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MAP PROJECTION

BY

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FOREWORD

This short class book makes no claim to go beyond the restriction imposed by its title—that of introducing the subject of map projection. It is written especially for candidates taking Geography in the H.S.C. Examinations and for first year University students. References in general class-books, when they are included at all, are necessarily too slight to be of much use for any specific study of the subject, while treatises on map projection, which usually would have to be passed round from the library, are generally too detailed to be readily grasped in the time available for a small section of the syllabus, or mathematically beyond the capacity of those candidates who ‘ drop ’ mathematics at the S.C. stage.

Though an attempt has been made to make the references to each projection complete in themselves, the whole is intended to make a graduated course directed towards the explanation of the principles on which such projection is based.

The writer expresses his indebtedness to Mr. G. Allman, M.A., and Mr. G. Kreisel for many valuable suggestions, and to the writers of those works on which consciously or unconsciously he may have drawn.

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SECTION ONE

THE PROBLEM OF MAP PROJECTION

I

THE GLOBE AND THE MAP

MAP PROJECTION is the representation, on a plane surface, of the meridians of longitude and parallels of latitude, reference to which defines any point on the earth's surface.

Mathematically, the term ' projection ', when restricted to the representation of points on a plane surface, means perspective projection ; i.e. the determining of the position on a plane of points in space as they would appear on the plane from a fixed view point.

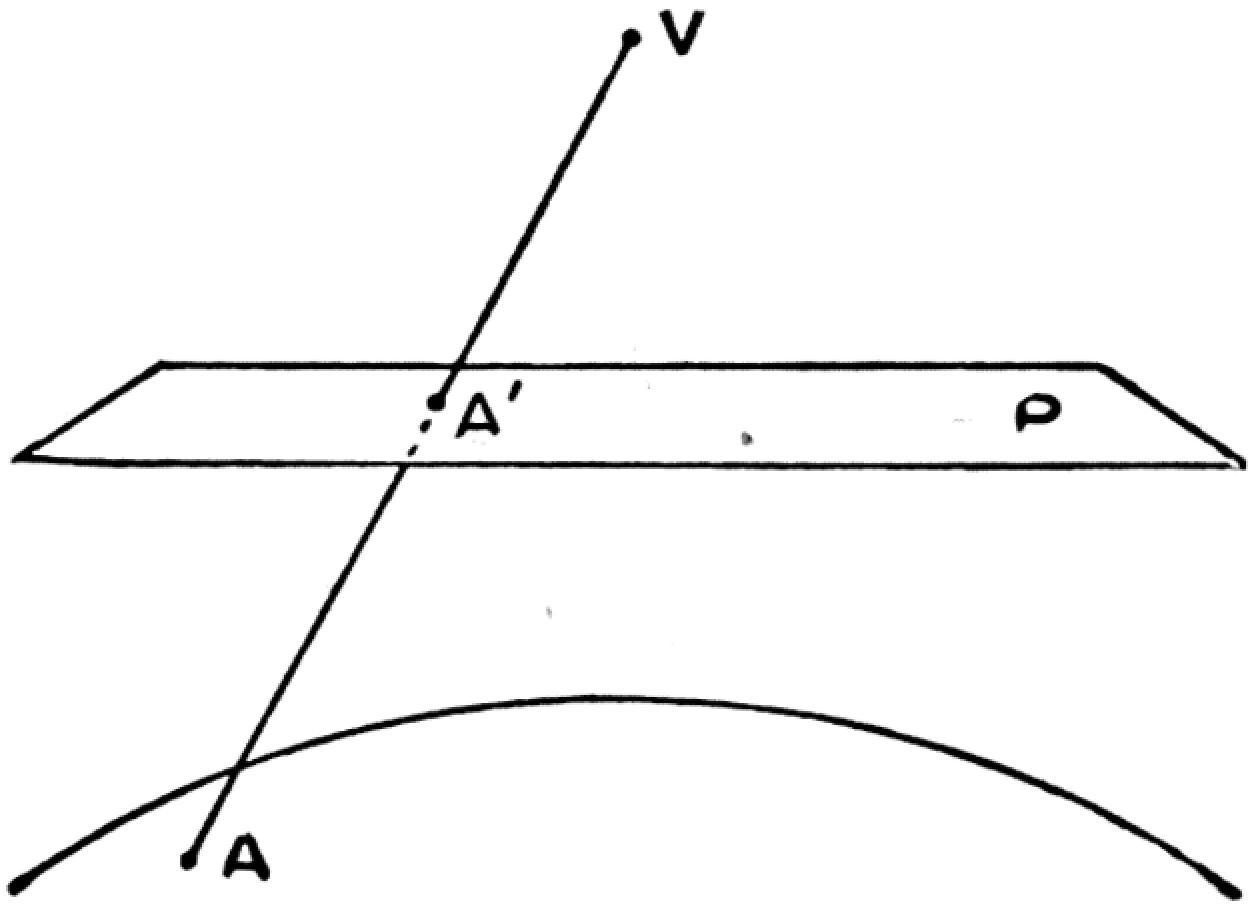


DIAGRAM I.—PROJECTION

In diagram I, *V* represents a view point and *A* a point in space (actually on the surface of a sphere). Imagine the plane *P* to be of glass. Then *A*, viewed from *V*, would appear on the plane at the point *A′*. In geometrical language, if a line *AV*, joining a point *A* and a view point *V*, cut a plane *P* in the point *A′*, then *A′* is the projection of *A* in the plane *P* from *V* as the centre of projection.

Unfortunately, few of the methods of map ' projection ' are projections in this mathematical sense and are simply arrangements of intersecting lines, sometimes curved, sometimes straight, to represent the meridians and parallels in accordance with some consistent principle selected to represent with the minimum distortion some aspect of the earth's surface. The resultant net-work of lines, the ' map net ', is often referred to as the GRATICULE (Latin *craticula* means ' a little hurdle '). Graticulation, strictly speaking, is the process whereby a drawing is reproduced on a different scale by the division of the surface into a number of squares which are then reproduced on the new scale.

Such precise similarity of reproduction is obviously impossible when one system of intersecting lines is spaced over a spherical surface and the representational system has to be made on a plane surface. The whole problem of map projection derives from the fact that the network of lines which form the meridians and parallels on the globe cannot be similarly represented on a plane surface ; nor can the relations between them on the globe be similarly represented on a flat map. One example should make this clear. On the globe the meridians intersect the Equator at right angles—yet they meet at the Poles. If the meridians are shown on a map as intersecting the Equator perpendicularly they will be parallel lines and will meet only at an infinite distance.

The interest in map projection lies in selecting a method of showing the meridians and parallels in such a way that some specific aspect of the globe is shown with the least possible distortion. Sometimes it is desirable to show areas proportionately, or to represent direction accurately, or to show distances or shapes with a reasonable degree of truth. Whichever particular quality or group of qualities is selected it is obvious that there will be corresponding distortion of the sacrificed qualities, especially if a large area of the globe has to be mapped.

Before beginning the study of specific projections, therefore, it is desirable to examine the meridians, parallels and their inter-relations as they appear on the globe. Throughout this book it is assumed that the earth is a sphere. On this assumption it is possible to regard the meridians as circles.

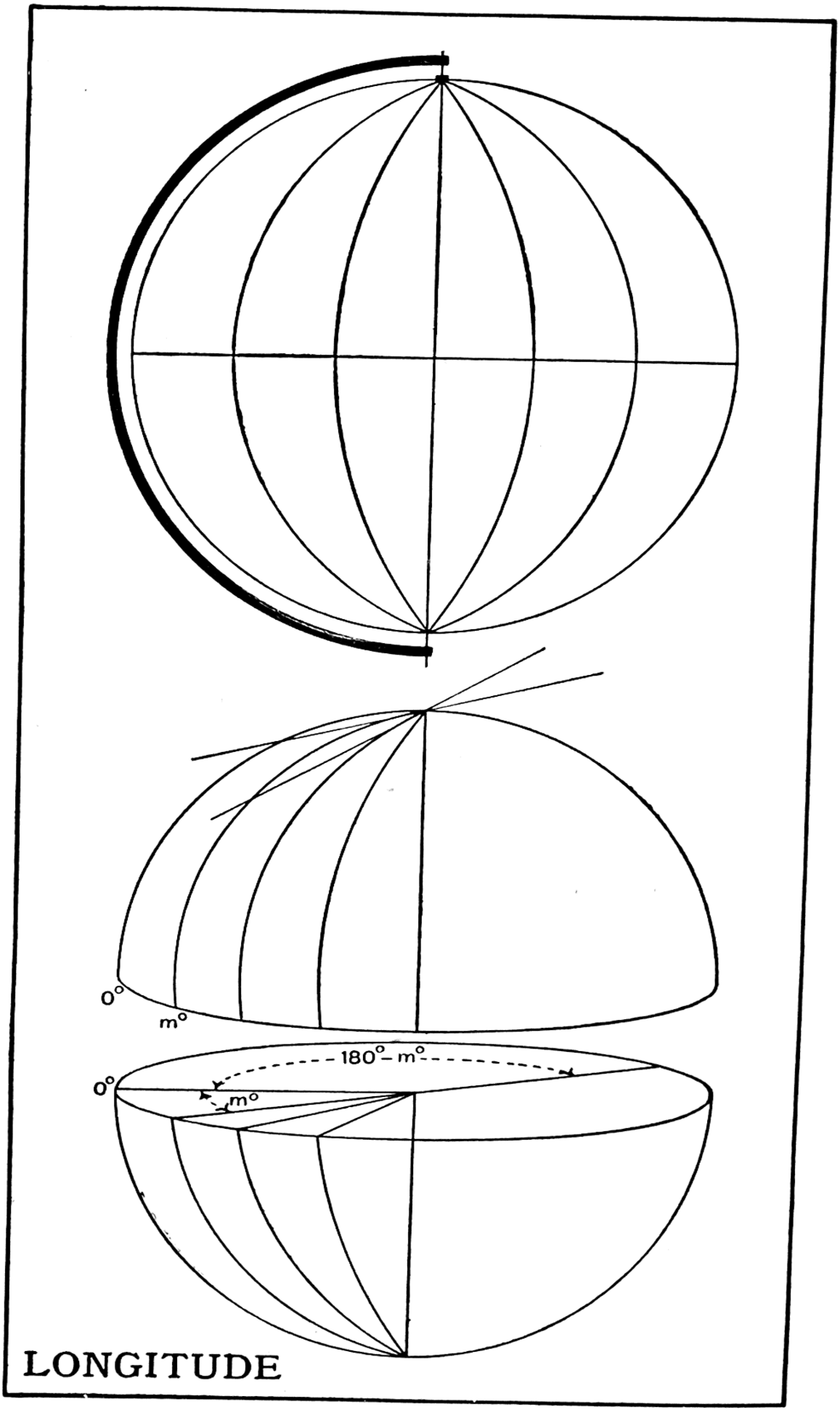


Diagram 2.—Meridians of Longitude

II

MERIDIANS AND PARALLELS

On a stationary or capriciously moving sphere there would be no point, other than some quite arbitrarily chosen one, from which one could begin to mark out the surface in any convenient way for reference to points on the sphere. The earth, however, by rotating, provides two usefully fixed points in the polar extremities of its axis of rotation. The EQUATOR is the circular line which, at equal distances from the NORTH and SOUTH POLES, divides the globe into hemispheres.

In order to divide the Equator into 360 equal parts, corresponding to the number of degrees in a circle, there is no fixed point from which one could begin ; consequently, such a point had to be arbitrarily chosen. Actually the point is out at sea, off the west coast of Africa, and is the point at which a circle through the Poles and through Greenwich would cut the Equator. Imagine a semicircular arc as being capable of swinging round a fixed diameter representing the polar axis of the earth, but actually at rest along and contiguous with the meridian which passes through Greenwich. The MERIDIANS are the positions the semicircular arc would occupy in passing through each degree from 1° to 180° east or west of the Greenwich meridian and are numbered accordingly.

There are, therefore, 180 complete circles, each passing through the Poles ; they divide the Equator into 360 equal parts and intersect it at right angles. Each circle has its centre at the centre of the earth, all the circles are equal, having as their radii the radius of the sphere, and the planes of the circles divide the globe into hemispheres. Circles which have these properties are known as GREAT CIRCLES. The meridians may be defined, therefore, as great circles which pass through the poles and which divide the Equator into 360 equal arcs and which make, therefore, by their intersections at the Poles, 360 angles of 1°. They divide the surface of the sphere into 360 equal LUNES (moon-shaped surfaces).

It follows that :

(*a*) the arc of the Equator between any two meridians subtends an angle at the centre of the earth equal to the difference between their respective degrees of longitude ;

(if the meridians are, say, 20° W. and 33° W. the angle subtended by the intersected arc would be 13°. If the meridians are $m°$ and $m'°$, the angle subtended would be $m° - m'°$) ;

(*b*) a similar angle would be made by tangents to these meridians at the Poles ; and

(*c*) opposite halves of a meridian are of supplementary degrees of longitude.

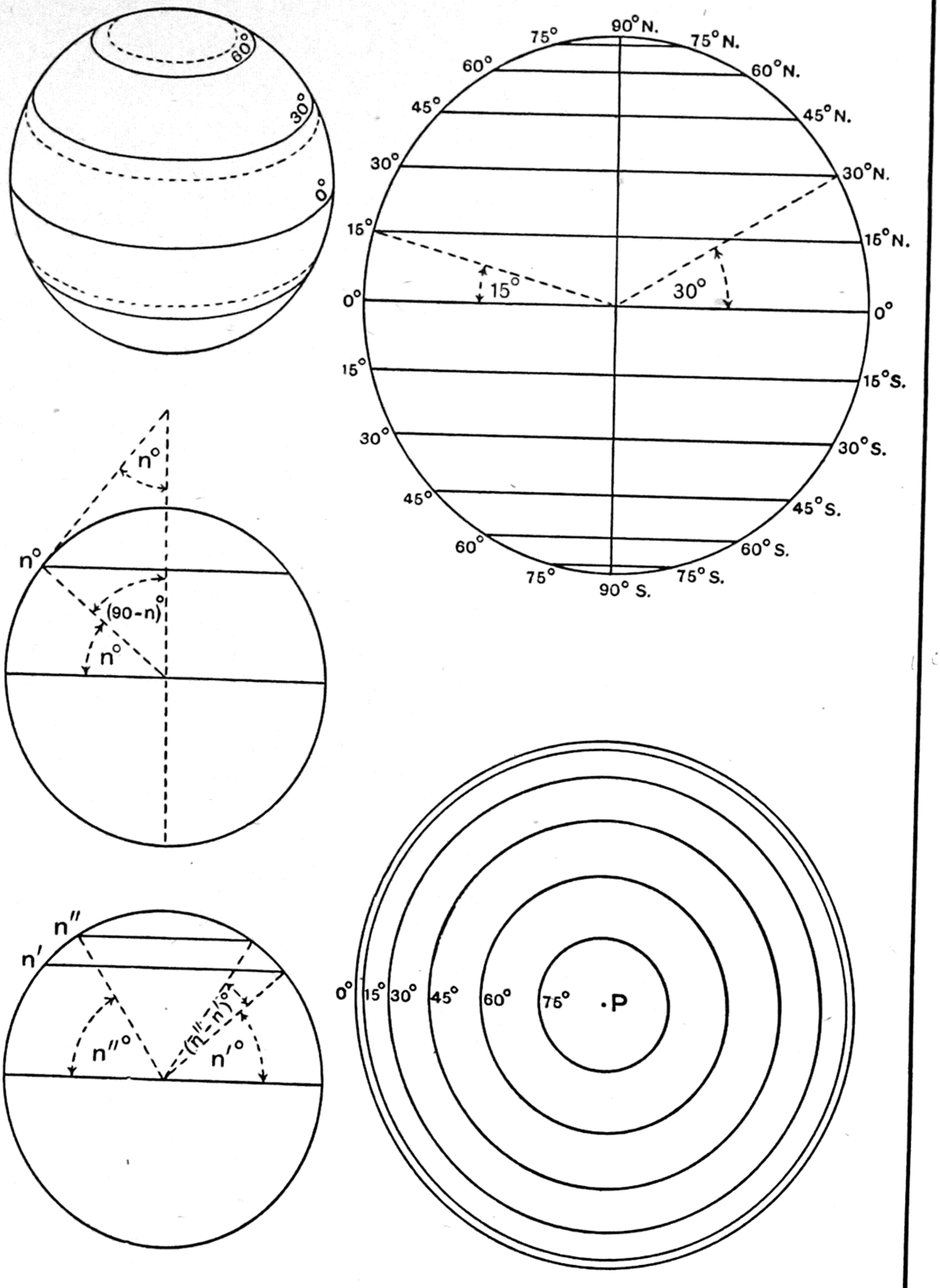
The PARALLELS OF LATITUDE on the globe are circles in parallel planes with their centres on the polar axis of the earth. Of the parallels only the Equator is a great circle. The other parallels decrease successively as they approach the Poles where they vanish in a point. They divide each quadrant of a meridian similarly into ninety equal arcs and, therefore, each such arc subtends an angle of 1° at the centre of the sphere. From the Equator to the Poles, north and south, the parallels are numbered from 0° to 90°.

It follows that :

(*a*) the arc of any meridian intersected by the Equator and a parallel of $n°$ of latitude will subtend an angle of $n°$ at the centre of the globe ;

(*b*) the arc between any two parallels n' and n'' will subtend an angle of $(n'' - n')°$ at the centre of the globe ; and

(*c*) the complementary arc between the parallel and the pole will subtend an angle of $(90 - n)°$ at the centre. This is called the angle of the co-latitude and in this book is usually called the angle $z°$.



PARALLELS OF LATITUDE

DIAGRAM 3

Certain RELATIONS BETWEEN THE MERIDIANS AND PARALLELS and some of the resultant problems will be clear at once from a glance at a globe.

(*a*) The intersection of every meridian and every parallel forms a right angle.

(*b*) the arcs of the meridians intersected by the parallels are all equal and are equal to the intersections of the Equator. Each such arc is, therefore, equal to a 360th of the earth's circumference, namely, just under 70 miles (actually 69·17 miles).

(*c*) the arcs of the parallels intersected by the meridians decrease from the Equator towards the Poles where they vanish.

The length of this arc at any given latitude is easily calculable. Firstly, what is the length of any given parallel in relation to the Equator? The circumferences of circles are related as are their radii, and from diagram 4 it is clear that parallel 60° is only half the length of the Equator.

(L represents the intersection of parallel 60° with the great circle $PLEQ$; angle EOL is, therefore, 60°.

LX represents the radius of parallel 60° and EO is the radius of the Equator.

By geometrical reasoning it is clear that $LX = \frac{1}{2}EO$;

$\therefore$ parallel 60° is half the length of the Equator.)

In general terms, if angle $EOL = n°$, then angle $OLX = n°$.

$\therefore LX = R \cos n°$, when R is the radius of the globe.

$$\therefore \frac{\text{any parallel } n°}{\text{Equator}} = \cos n°.$$

We noticed that parallel 60° is half the length of the Equator. Cos $60° = \frac{1}{2}$. Since the arcs intersected by the meridians will be proportionately reduced, it follows that the arcs of parallel 60° intersected by the meridians will be just under 35 miles. It is an easy matter to work out similar examples.

Example : What is the approximate distance in miles along parallel 40° between meridians 8° W. and 0° (the distance across central Spain)?

Distance between two consecutive meridians, 40° N. or S., is

$$\cos 40 \times 69 \text{ miles} \simeq \cdot 766 \times 69 \text{ miles} \simeq 53 \text{ miles}.$$

$$\therefore \text{required distance} \simeq 8 \times 53 \text{ miles} = 424 \text{ miles (approx.)}.$$

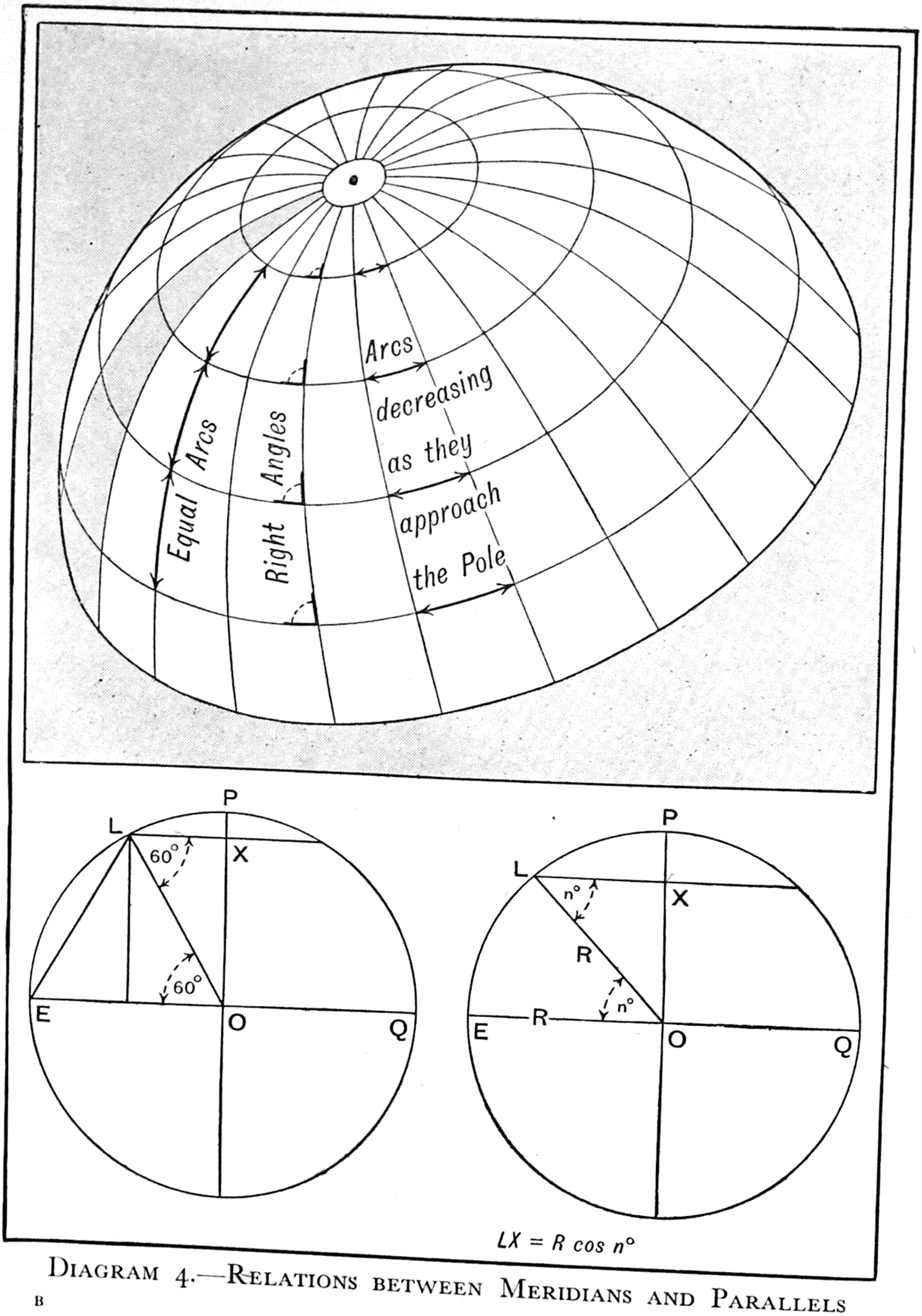
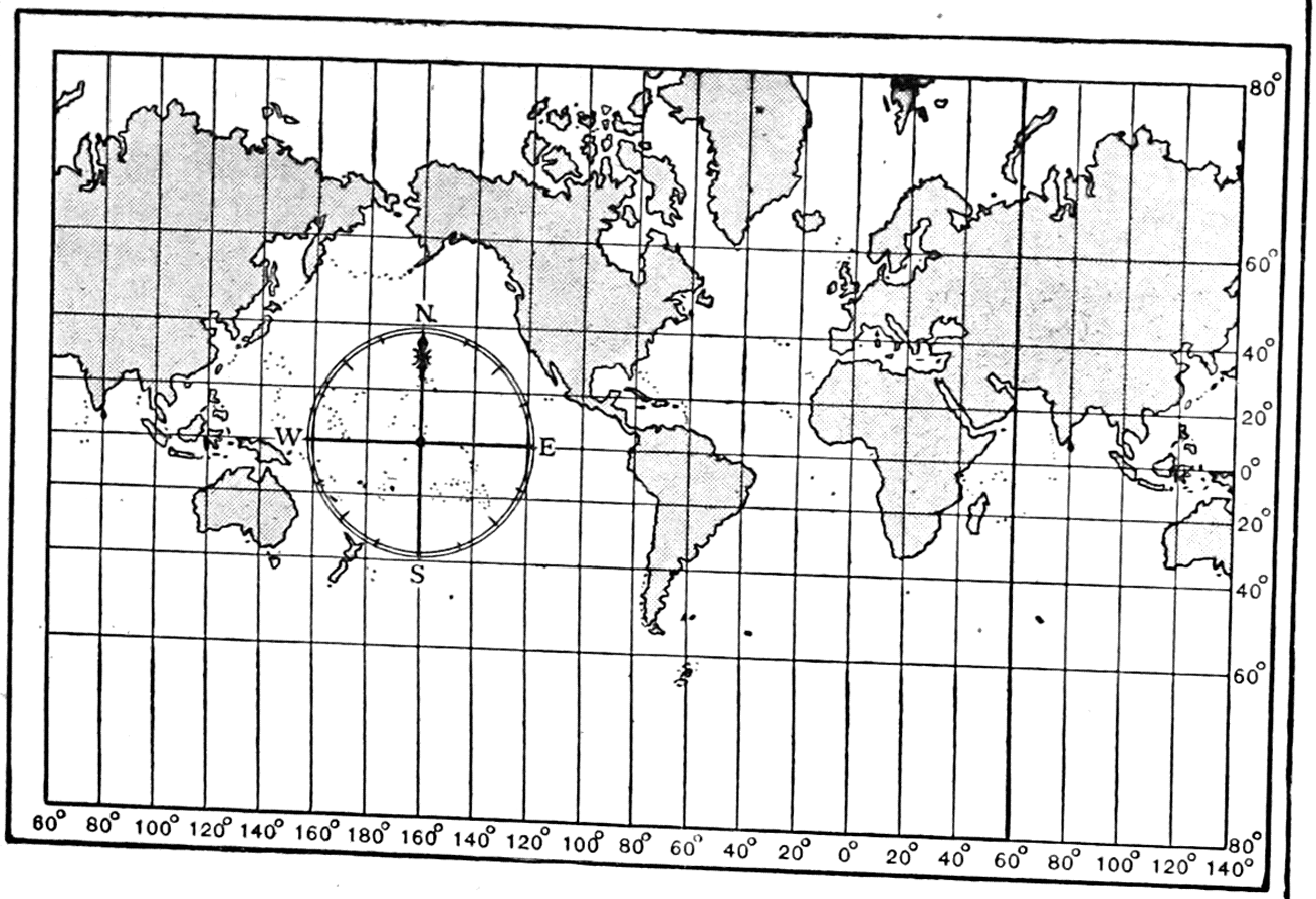


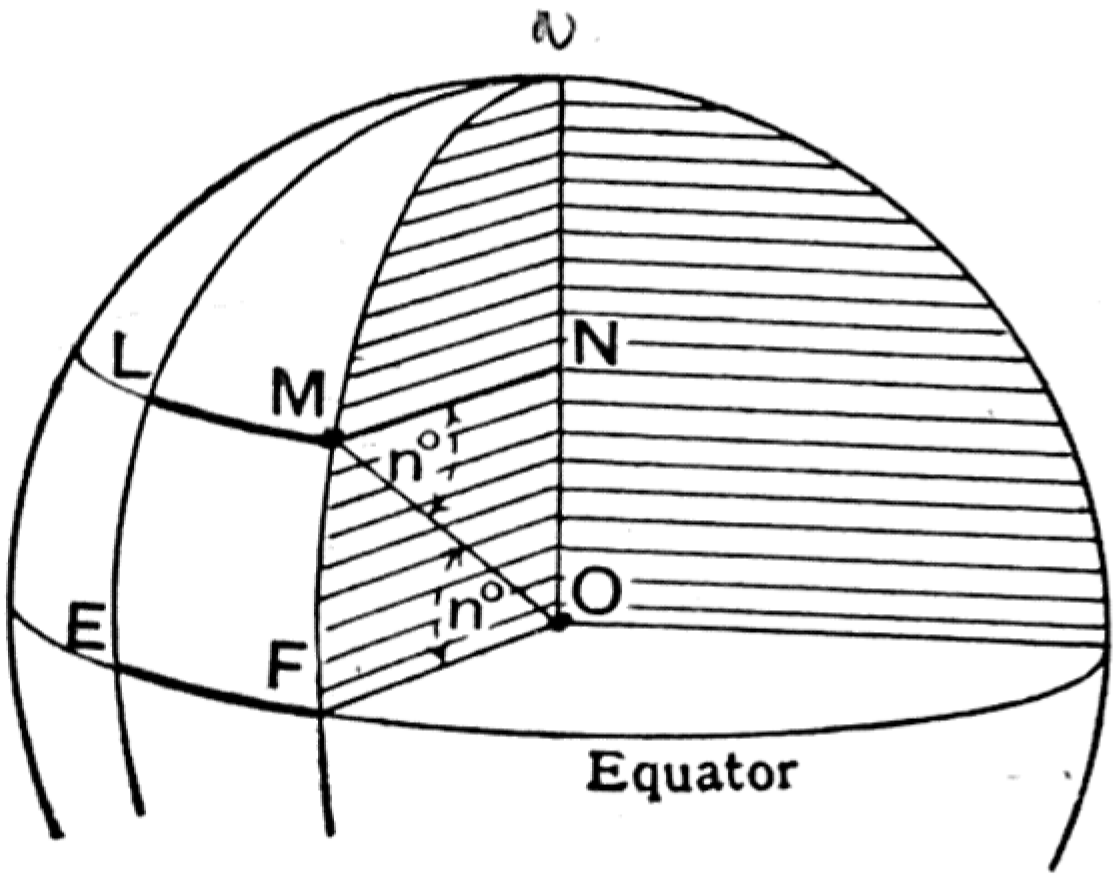
DIAGRAM 4.—RELATIONS BETWEEN MERIDIANS AND PARALLELS

Imagine a map which is designed to retain the first of the three relations we have noticed, by representing the meridians and parallels as straight lines intersecting at right angles, and let us assume that it is a map of the world. Such a projection is that known as Mercator's (see page 11). The Equator is divided into 360 equal parts and the meridians are drawn perpendicularly through these points. As the meridians will be parallel it is immediately obvious that such a map cannot show the Poles. Moreover, the distance between the meridians will be constant and the parallels will be represented by *equal* straight lines. Parallel 60° and, therefore, divisions along parallel 60° will be twice their true length. Distances along parallel 80° will be over five and a half times their true length!

Obviously the true lengths can be shown along the meridians, but if it is required to keep the correct proportions between the meridians and parallels, obviously the distances along the meridians, the distances between the parallels, will have to be proportionately increased. This is done in Mercator's Projection. For example, as the distance between any two meridians at parallel 60° is twice as great as it should be, in order to preserve the proportion between the meridians and parallels one degree of longitude at parallel 60° must be doubled. Thus, as one approaches the polar regions, both distances and areas are increasingly exaggerated. No one would imagine, for example, from a map on this projection, that Greenland is less than half the size of the Indian Empire! (Greenland has an area of 830,000 square miles and India with Burma an area of 1,800,000 square miles.)

It may be wondered of what value a map so distorted could be. Actually, Mercator's Projection has an important use in that all compass directions, such as North-West, South-West by West, and so on, are accurately indicated by straight lines drawn between any two points. It is at once apparent that, in order that a specific purpose should be achieved by a map, a very considerable degree of distortion in other respects may have to be tolerated.





(a)

Let m° of longitude be represented by EF at the Equator, and LM at n° of latitude.

Then, $\frac{LM}{EF} = \frac{MN}{FO} = \frac{MN}{R} = \cos n^\circ$

$\therefore LM = EF \cos n^\circ$

i.e. a degree of longitude at n° latitude = a degree of longitude at the Equator × cos n°.

MERCATOR'S PROJECTION

DIAGRAM 5

III

DIRECTION, DISTANCE, AREA AND SHAPE

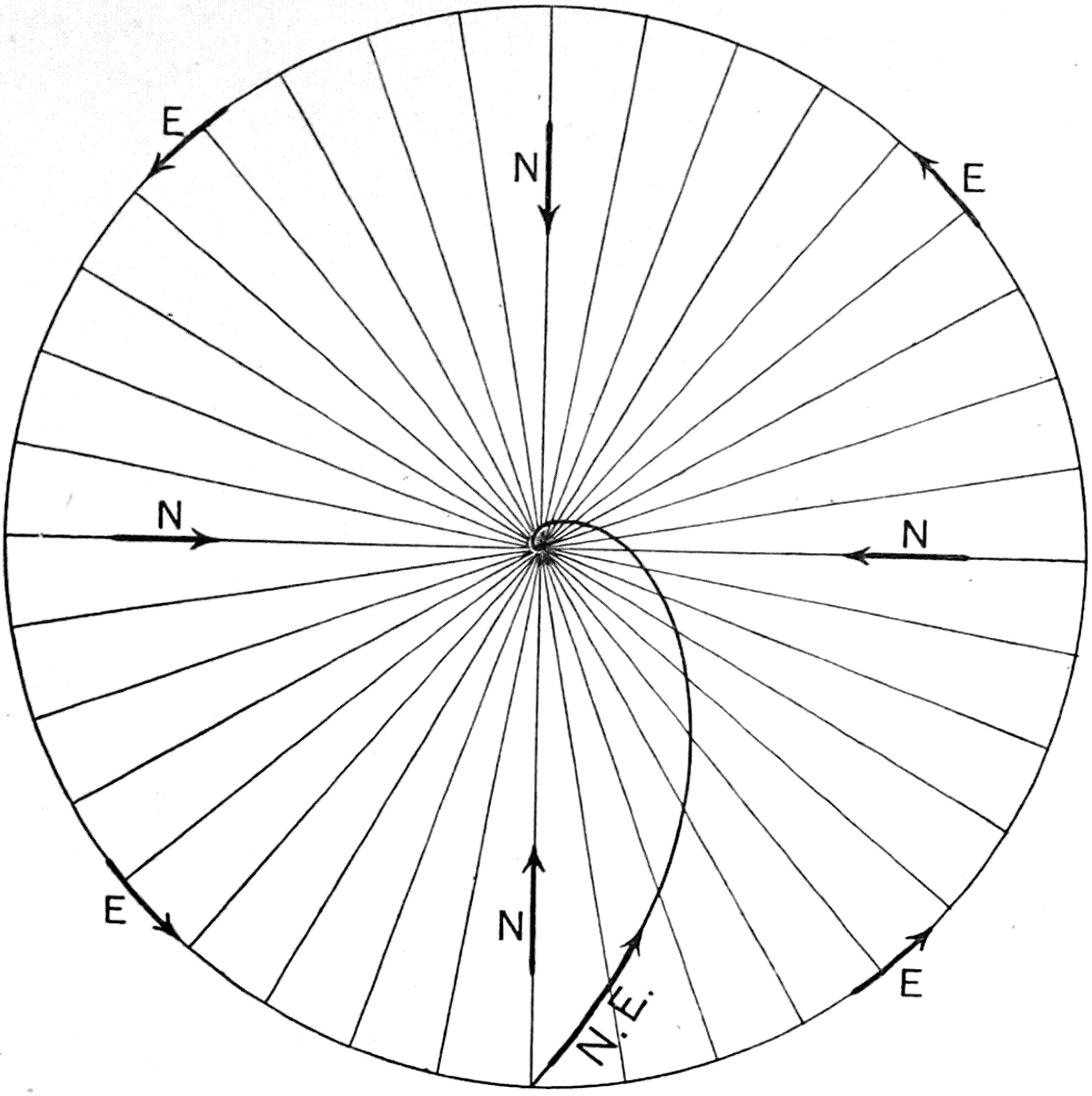
Direction. Until we have become accustomed to thinking of *direction* in relation to the globe, the concept can have quite a deceptive quality. Imagine two points moving due north from any two positions on the Equator :

(1) Their loci would be at right angles to the Equator ;
(2) Their loci would intersect every parallel at right angles ;
(3) They would, however, meet at the North Pole ;
(4) At any moment until they reached the Pole a line due east would mean a line perpendicular to their respective paths ;
(5) Such a line would intersect every meridian at right angles.

It follows that a line N.E. from any given point is a line which, passing through that point, intersects every meridian at an angle of 45°. Any such line, representing a compass direction from any given point, is known as a RHUMB LINE (sometimes as a RUMB LINE), and it may be defined as *a line which cuts all the meridians at the same angle.*

What would be the locus of a point moving, say, from the Equator in a consistently north-easterly direction? Constantly reaching a position nearer and nearer to the North Pole, it would approach it spirally, along a track known as a LOXODROMIC CURVE. The term ‘ loxodromic ’ means simply ‘ in an oblique course ’.

For navigation by the compass or ‘ by the rhumb line ’ a map which shows accurately every compass direction between any two points as a straight line would be extremely useful—and Mercator’s does this. It does it quite obviously, because the perpendicular intersections of parallels and meridians and the proportionate distances along each are both preserved ; consequently, any line which cuts the meridians on the globe at a constant angle will be represented in Mercator’s Projection by a straight line which cuts the meridians at the same angle and in similar places.



Locus of a point moving consistently so that it crosses the line representing every tenth meridian on a polar projection of the northern hemisphere.

Diagram 6.—Loxodromic Curve

Certain projections show the true bearings or AZIMUTHS of all points from the centre of the map. Such projections are known as AZIMUTHAL. If the globe were viewed, as in true perspective projection, from some point ‘ overhead ’, it is clear that all points would appear in their true direction from this view point, which would itself be projected as the central point of the map. Such projections are known as ZENITHAL projections, for ‘ zenithal ’ means, literally, ‘ overhead ’. It follows that *all zenithal projections are azimuthal.*

Distance. It is often said that a ship which sails 'in the same tack', i.e. in a fixed direction, by rhumb line, is taking 'the straight course'. Actually it is doing nothing of the kind unless it is sailing due north or south or along the Equator. Even if it is sailing due east or west, along a parallel of latitude, it is not sailing 'in a straight line' or, in other words, taking the shortest route.

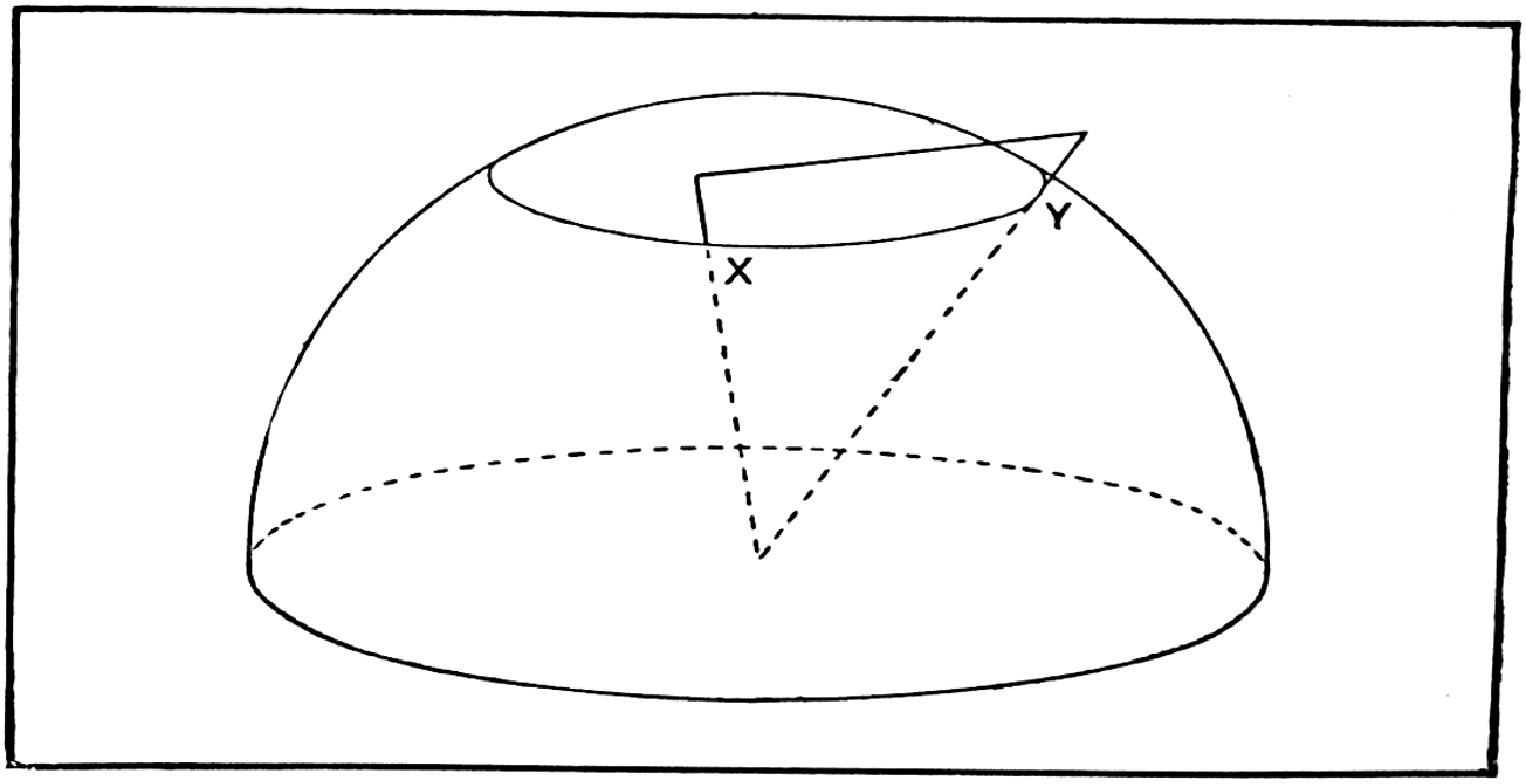


DIAGRAM 7.—'THE STRAIGHT COURSE'

Imagine two vertical poles, i.e. two poles which would pass through the centre of the earth, sufficiently high to allow a strong cord to be suspended rigidly between them for many miles, both ends on the same parallel, say from the English Channel to the Gulf of St. Lawrence. A ship, taking the shortest possible course and what might reasonably be called a straight course, would obviously sail under the suspended cord; *but it would not be sailing along parallel 50° nor would it be sailing due west,* although the Gulf of St. Lawrence is due west of the English Channel. A glance at the globe will show that the ship would begin by sailing north-west by west, leaving the parallel 'to the port side'; after a time, with the parallel now well to the south, it would be sailing momentarily westwards before its compass would begin to register south-west by west, until finally it would meet the parallel again at Newfoundland.

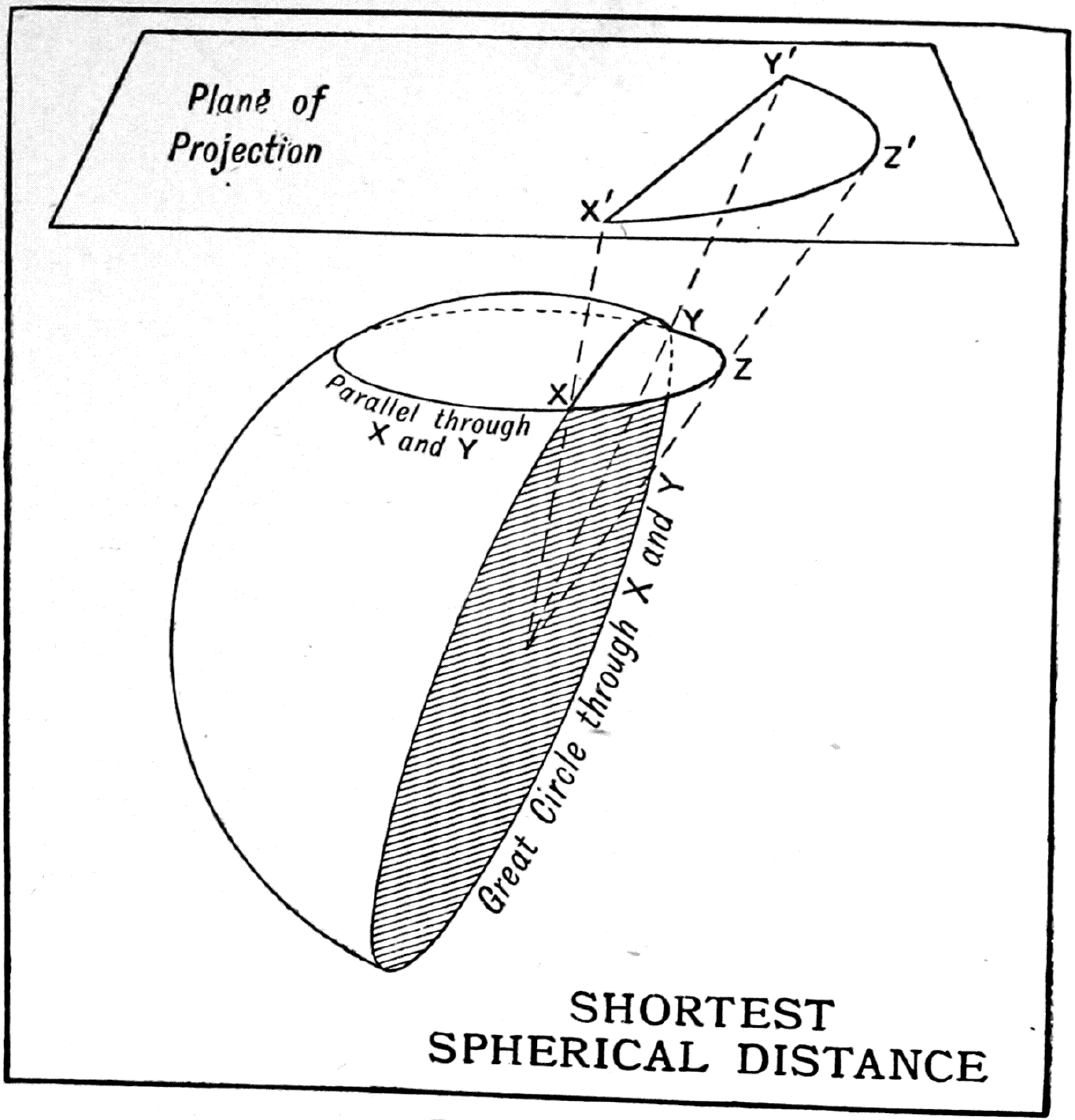
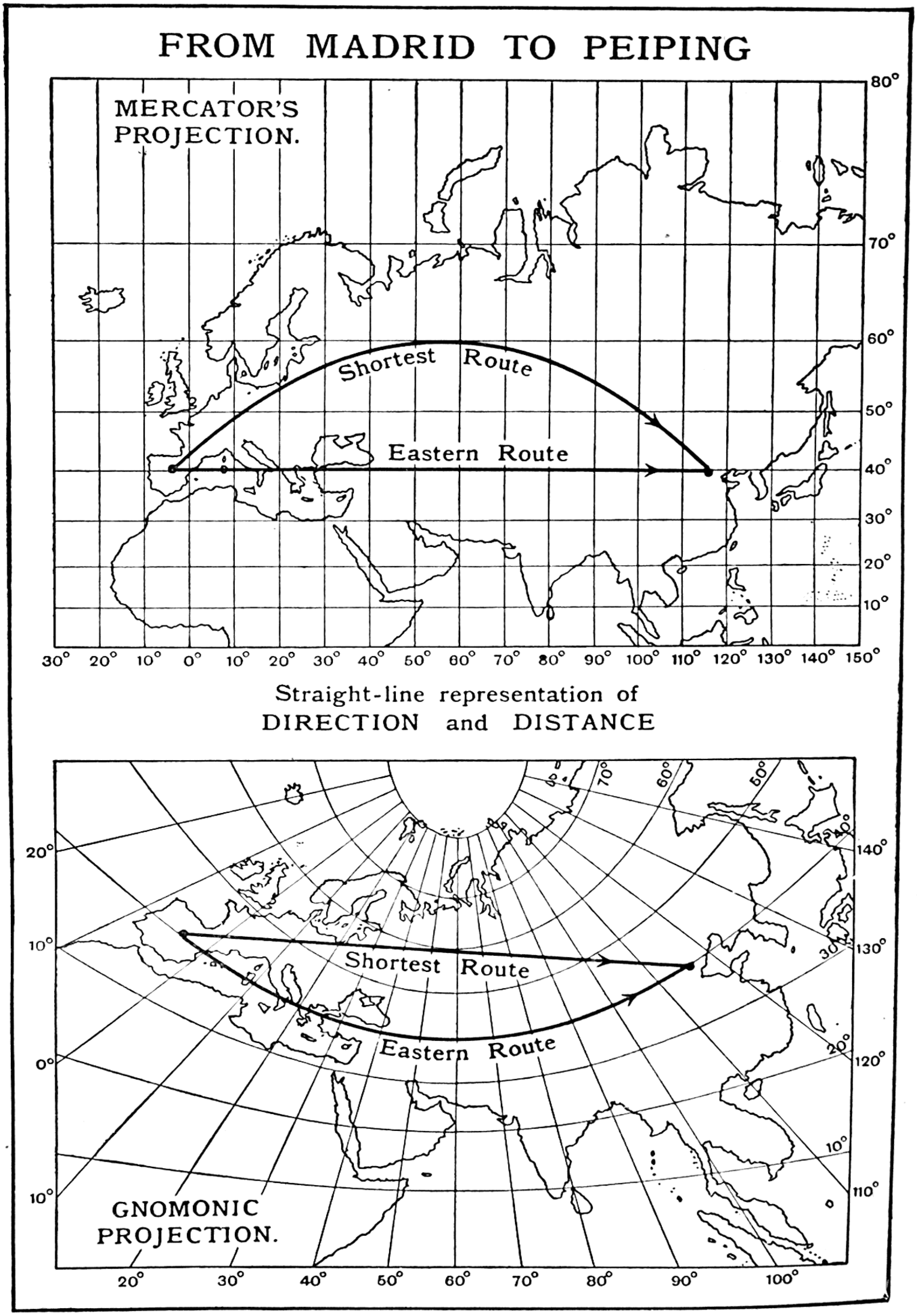


DIAGRAM 8

The shortest spherical distance between two points is the shorter of the two arcs which they cut off from the great circle on which they lie. Imagine—or make—any two marks on a soft ball. Wherever these points are it is obviously possible to cut the ball through these points into two hemispheres, so that the points will lie on the circumference of the sectional circle. The shorter of the two arcs between the two points will be the shortest spherical distance between them, because, viewed from the centre of the sphere (when the eye would be in the same plane as the sectional circle) that distance would appear as a straight line, corresponding to the cord suspended over the North Atlantic. In other words, the projection of a great circle, viewed from the centre of the sphere, is a straight line (see diagrams 7 and 8).



FROM MADRID TO PEIPING

Straight-line representation of DIRECTION and DISTANCE

DIAGRAM 9.—DIFFERENCE BETWEEN STRAIGHT-LINE REPRESENTATION OF DIRECTION AND OF DISTANCE

For ' shortest route ' or ' great circle ' navigation a map showing great circles by straight lines would obviously be useful, and it should already be apparent that such a map would be produced by the projection of the meridians and parallels from a view point situated at the earth's centre. This follows from the fact that from such a point, and from such a point only, all great circles would lie in the same plane as the view point, and the section of a plane is a straight line. This projection, which is described later (page 45), is known as the Gnomonic or Central Projection. Diagram 9 shows two maps of the same region, one a section of a Mercator's Projection of the world, in which compass directions are shown as straight lines, and the other a Gnomonic Projection in which great circles are shown as straight lines. The difference between rhumb line and great circle navigation should be at once apparent.

Just as there are maps which show directions correctly from the central point of the map, the azimuthal or zenithal projections, so there are projections which show distances correctly from the central point. Such projections are known as Equidistant.

Equal Area or Equivalent Projection. It is often specially desirable that a projection should show accurately relative areas, as in statistical or distributional maps.

If the meridians and parallels are represented as intersecting at right angles, and if distances along them could be perfectly represented, it follows that the map would be an equal area map, for the product of the lengths of a section of a meridian and of a parallel from the point of intersection would agree, when made from the map, with the product of corresponding distances on the globe. This perfection is impossible on a plane surface ; but obviously the correct *product* can be preserved by adjusting the scale of distances along the meridians and parallels suitably.

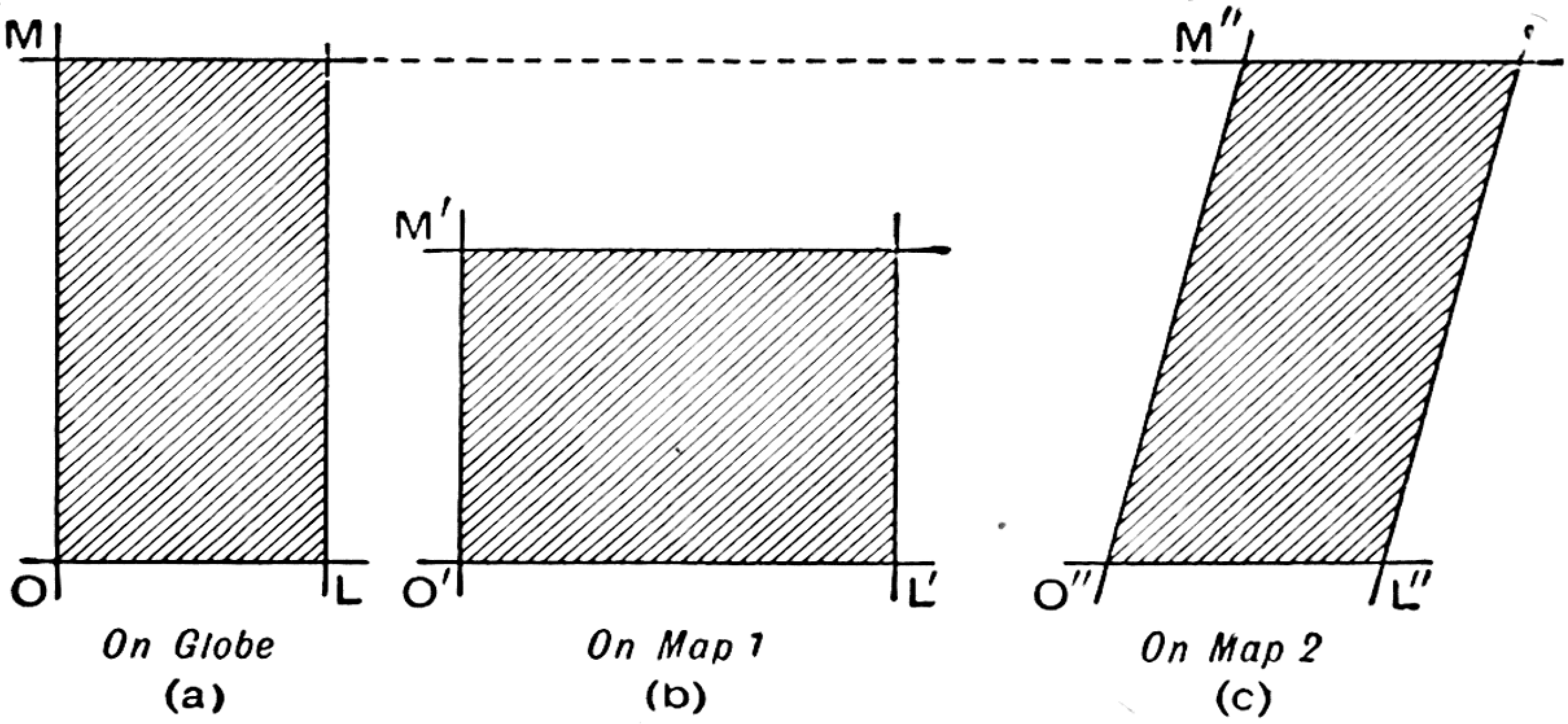


DIAGRAM 10.—HOW EQUALITY OF AREA MAY BE RETAINED

For example, if the meridians and parallels are shown as straight parallel lines intersecting at right angles, the distance between the meridians at any latitude remains constant, from the Equator to the Poles, whereas on the globe it decreases until it vanishes at the Poles. To preserve equality of area on such a map the distance between the parallels must be proportionately decreased towards the Poles, as in diagram 10 *b*.

When the meridians and parallels are not shown as intersecting at right angles, the perpendicular distance, from a parallel, of any point on a meridian, or vice versa, must be similarly adjusted to give the required product in order to preserve equality of area. (Diagram 10 *c*.)

Shape. Finally, it is desirable always that there should be as little distortion of shape as possible. No map, however, can reproduce correctly the shapes of all places as they appear on the globe. The term ORTHOMORPHIC, which means ' correctly shaped ', cannot, therefore, mean quite what it says. The name is an unfortunate one for so-called orthomorphic maps have a special significance and may not always represent shapes as accurately as other projections which are not orthomorphic. *A projection is orthomorphic when the scale along the meridians and parallels is the same in both directions and when the meridians and parallels are at right angles.* It should already be clear that Mercator's Projection is an orthomorphic one. In all such projections shapes are correctly shown for small areas, but for small areas only. That small areas are correctly shown is because the angles in any small figure on the globe will remain unchanged on the map, as the scale is the same in all directions. That it is true for small areas only is obvious from Mercator; for the continuous increase in all dimensions to the north and south of the Equator implies that, in any large area, the part nearer to the Poles is represented on a greater scale than the part nearer to the Equator, a fact which implies some distortion of shape. As, by definition, all orthomorphic projections represent the meridians and parallels at right angles, this degree of distortion must be a characteristic of all of them.

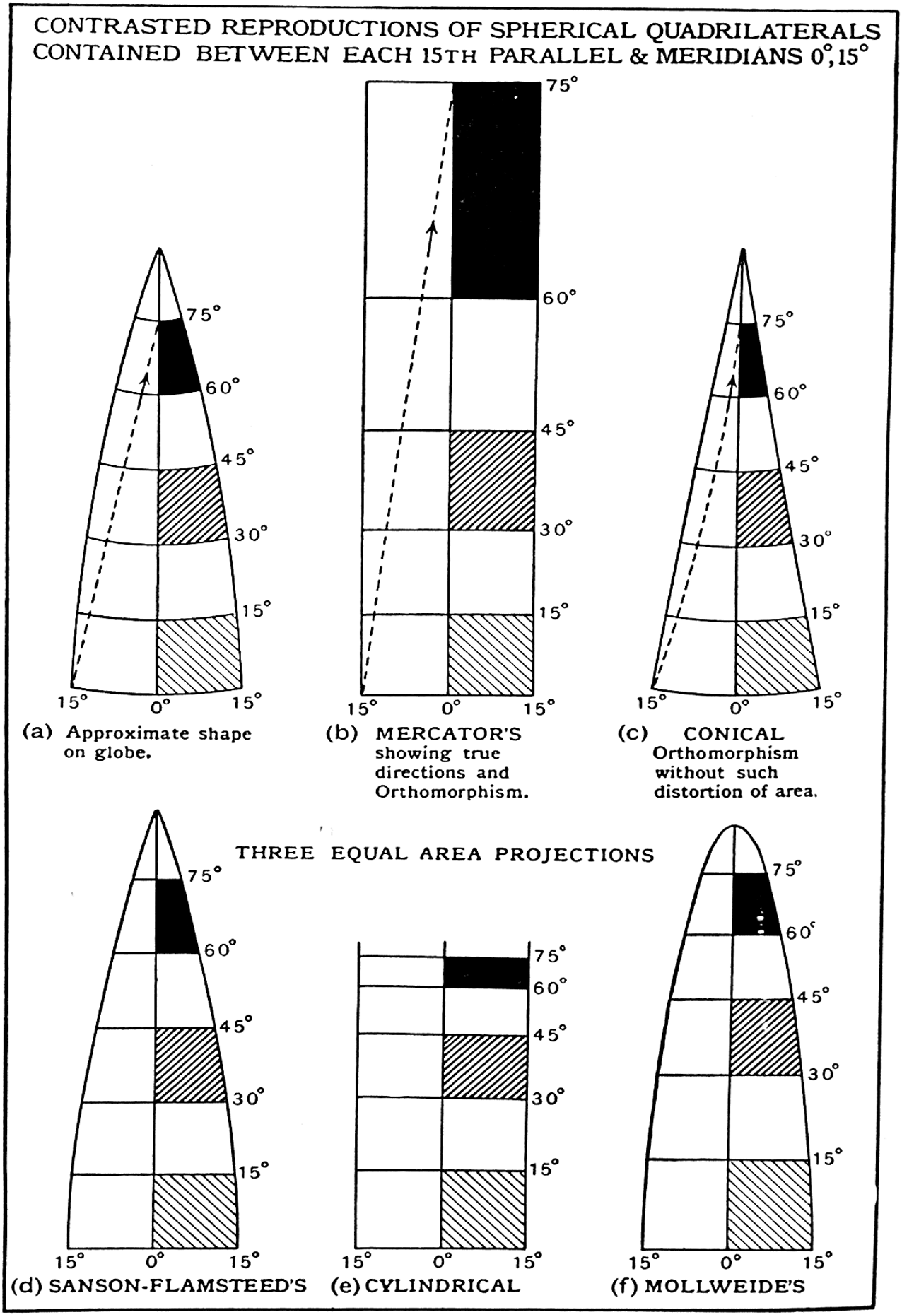

CONTRASTED REPRODUCTIONS OF SPHERICAL QUADRILATERALS
CONTAINED BETWEEN EACH 15TH PARALLEL & MERIDIANS 0°, 15°
75°
60°
45°
30°
15°
15°
0°
15°
(a) Approximate shape on globe.
(b) MERCATOR'S showing true directions and Orthomorphism.
(c) CONICAL Orthomorphism without such distortion of area.
THREE EQUAL AREA PROJECTIONS
(d) SANSON-FLAMSTEED'S
(e) CYLINDRICAL
(f) MOLLWEIDE'S

Diagram 11.—Contrasted reproductions of Spherical Quadrilaterals in Different Classes of Projection

Classification of Map Projections in relation to their Purpose. It is now possible to classify map projections in relation to the main or specific object they are intended to achieve. The problem is simplified if one thinks of it in terms of the four-sided figure formed by the intersections of two parallels and two meridians. The figure is rectangular on the globe yet two opposite sides are unequal, a combination of qualities which cannot be reproduced at the same time on a plane surface by straight lines.

On the globe two sides of this spherical quadrilateral, the sections cut off along the meridians by the parallels, remain of constant length ; the other two sides decrease proportionately, as we have already seen, so that their relation to the meridional arcs is that of cosine latitude. If the map is required to show all *directions* accurately the projection must preserve the rectangular quality of the spherical quadrilaterals and their linear similarity, that is, the correct proportions of length and breadth, so that the rhumb line would intersect all meridians at the required angle and in places similar to its intersections with the meridians on the globe. If it is desired to show correctly *all directions from a central point*, i.e. to construct an *azimuthal projection*, all that is necessary is to represent the net work of spherical quadrilaterals as they would appear when viewed from a point vertically over that central point ; i.e. to construct a zenithal projection. If it is desired to show all *distances* correctly as straight lines, then the map has to be projected on to a plane from a view point situated at the earth's centre. An *equal area* map implies that, however distorted in shape each spherical quadrilateral may appear in the projection, the area enclosed by the four arcs must be correctly represented by the corresponding four-sided figure in the projection. When the general principle of reasonable accuracy of shape is super-imposed on these specific ends, it follows obviously that the shape of the relevant spherical quadrilaterals which cover a particular region has to be preserved as far as is practicable. Owing to the size of the earth it is possible to make a reasonable compromise between all these four aims when planning a map of a relatively small region. A glance through any good atlas will show the diversity of ways in which these various objects have been attempted, what sacrifices have been made in order to achieve a specific object, and what problems have been left unsolved in the mapping of large areas.

The following section is an analysis of the main projections in use and a description of the methods by which they may be constructed.

SECTION TWO

THE PROCESS OF MAP PROJECTION

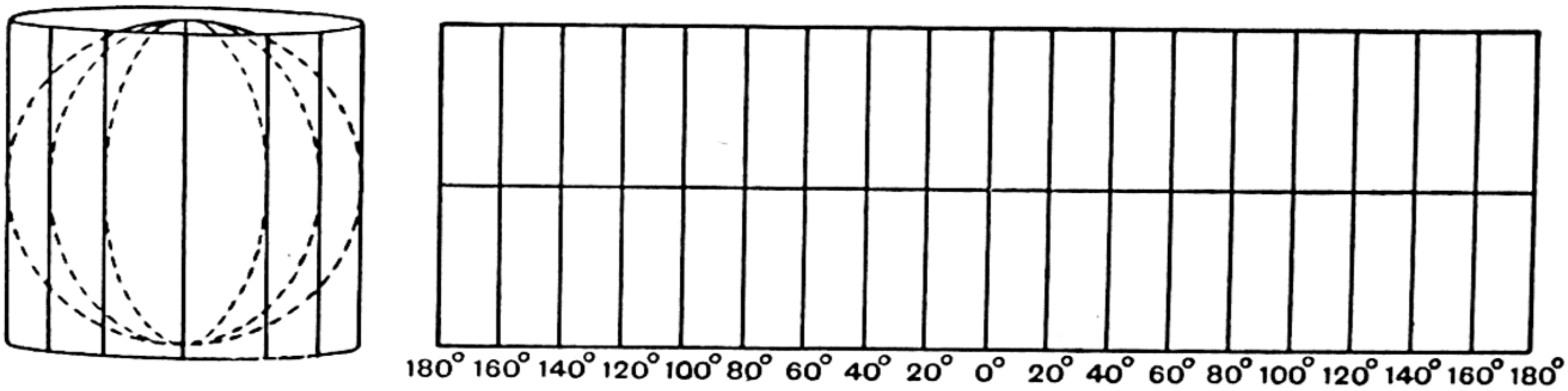
I

CLASSIFICATION OF MAP PROJECTIONS IN RELATION TO METHOD OF CONSTRUCTION

THERE are three important systems of map projection as classified in relation to method of construction, known respectively, and rather unfortunately perhaps, as *Cylindrical*, *Conical*, and *Zenithal*. It is possible to regard them all as varieties of conical projection.

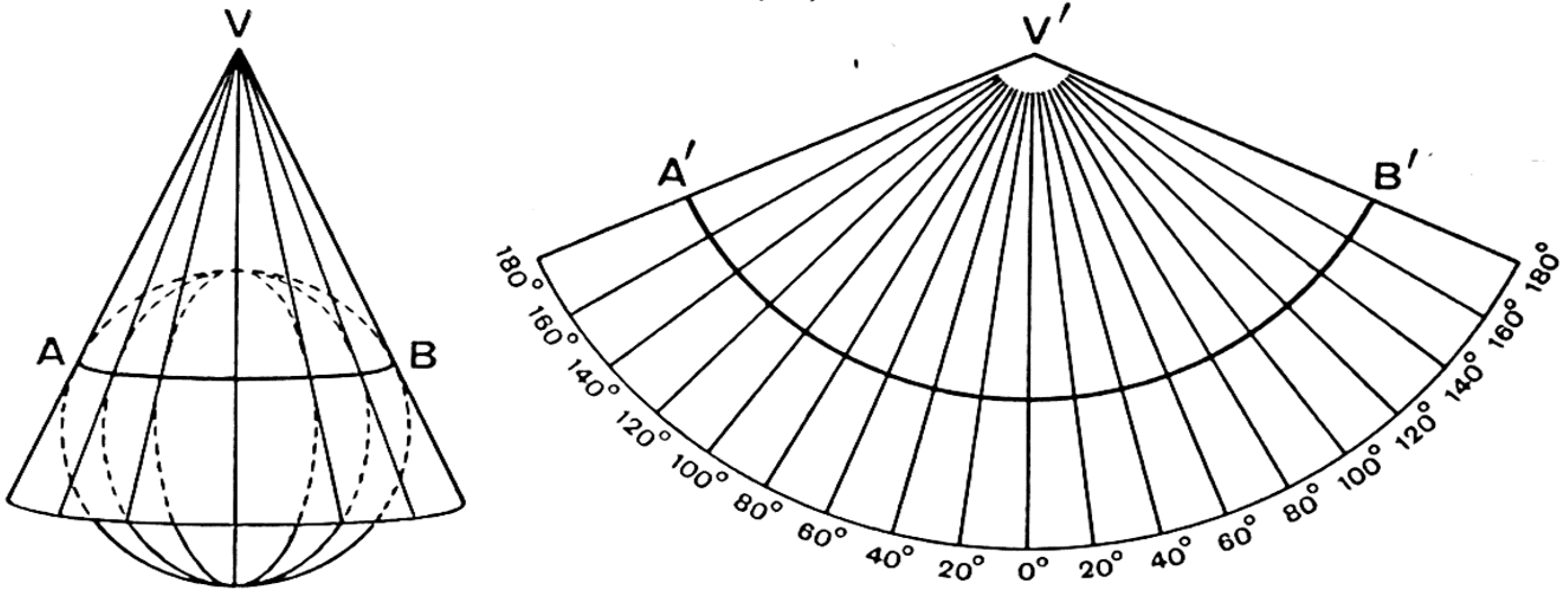
Imagine a cone as being tangent to the globe at any selected parallel as in diagram 12 (*b*). On this cone's surface, which is capable of being developed, or laid out flat, the meridians and the selected or 'standard' parallel are to be projected. The nearer the selected parallel is to the Equator the more remote will be the vertex of the cone, until, when the Equator itself is the selected parallel, the vertex will be infinitely remote. The globe will then be circumscribed by a tangent cylinder on the surface of which the meridians and the Equator are to be projected as in diagram 12 (*a*). Conversely, the nearer the selected parallel is to a Pole the nearer will the vertex of the cone be to the globe, until when the selected parallel is the Pole itself, when the 'cone' touches the globe only at a Pole, the vertex will also reach the Pole. The cone will have been completely flattened out and have become a tangent plane. It is convenient, however, to treat the three classes separately, though it is useful to remember that they are really interdependent systems.

Cylindrical Projection may, then, be defined as that class of projections in which the meridians and the Equator are conceived as being projected on to the surface of a cylinder which circumscribes the globe and is tangent to it at the Equator. The surface of the cylinder, when unfolded, will reveal the meridians as vertical and parallel straight lines, intersecting the Equator at right angles and dividing it into 360 equal parts. The parallels will be represented by horizontal straight lines at some arbitrarily selected distance from the Equator



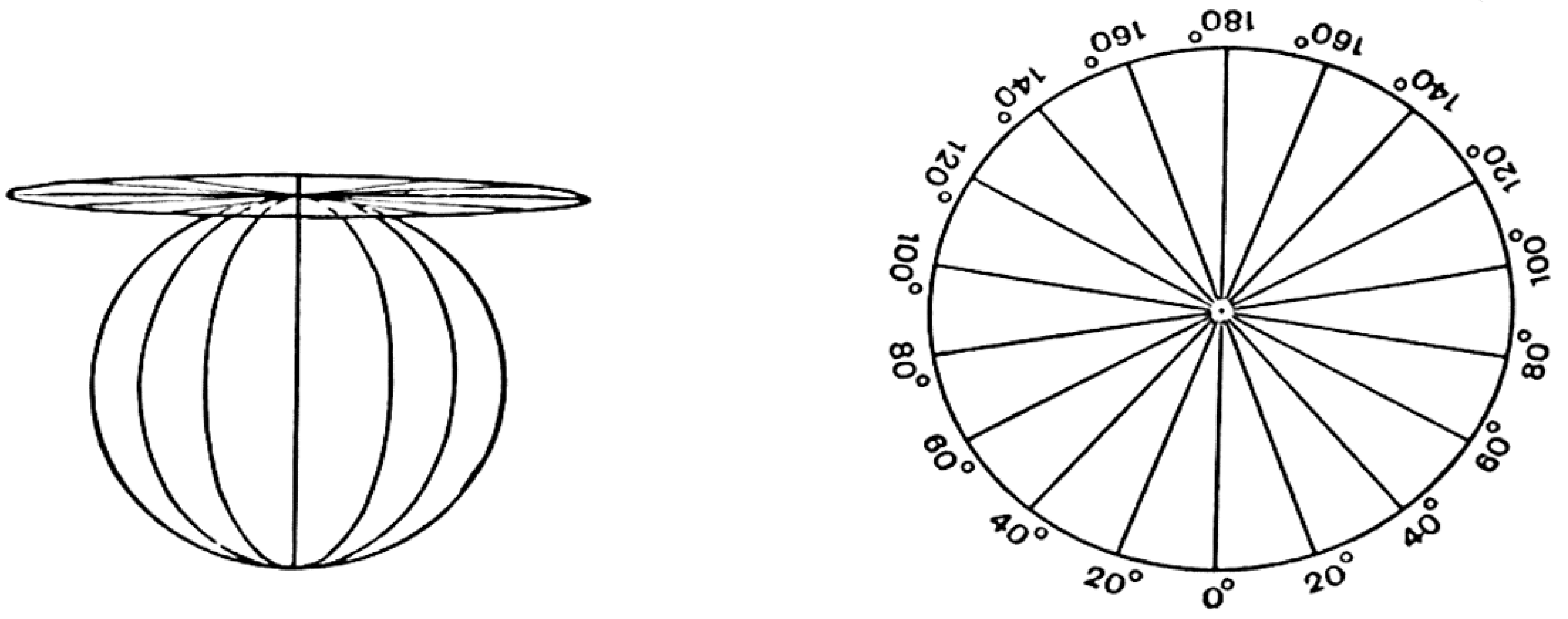
CYLINDRICAL

(a)



CONICAL

(b)



ZENITHAL

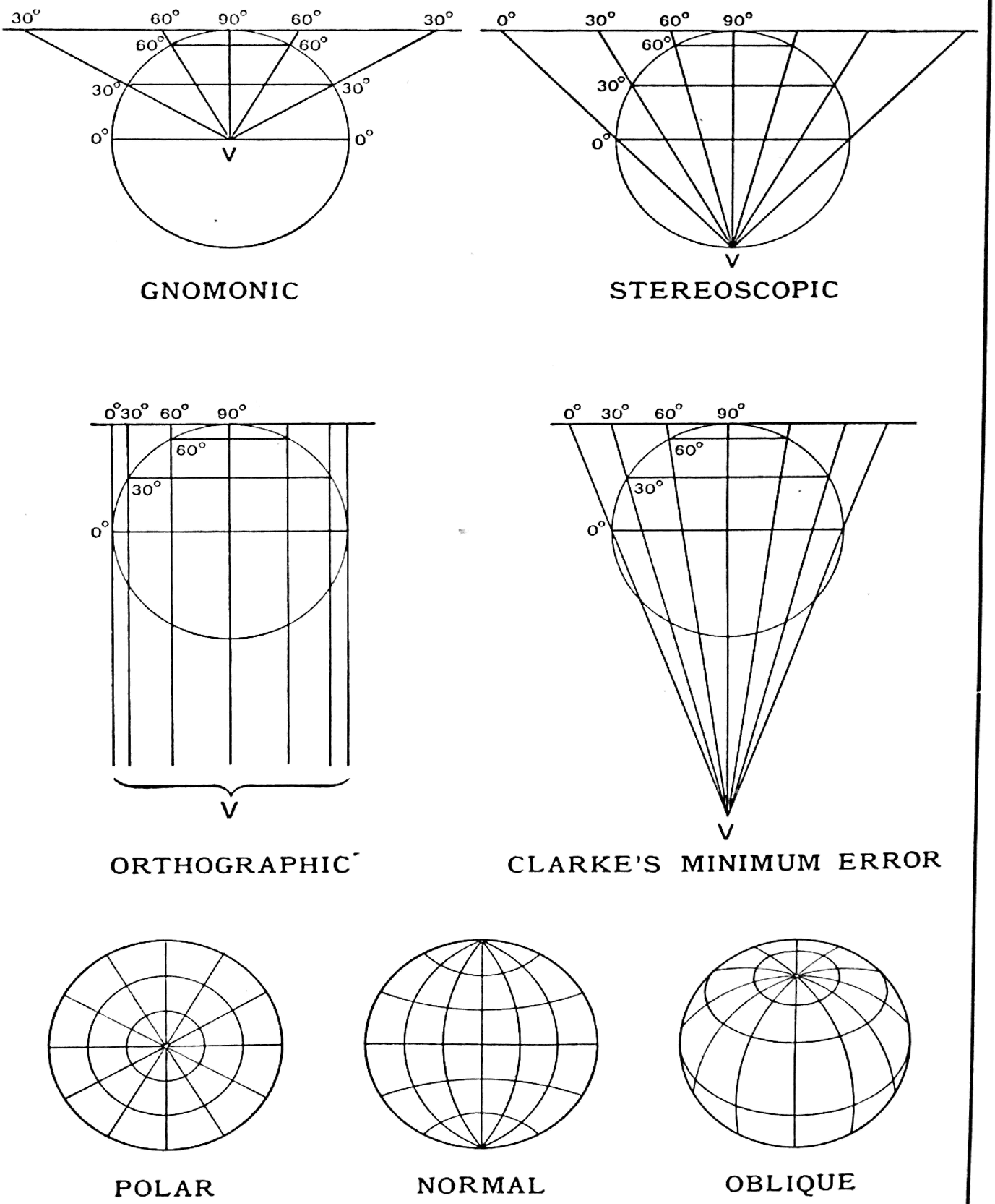
(c)

DIAGRAM 12.—CYLINDRICAL, CONICAL AND ZENITHAL PROJECTIONS and from each other. The spacing of the parallels in this and all the projections is arbitrary only in the sense that it is not necessarily dependent on the projection but on some mathematical principle selected or designed to satisfy best the conditions the map is intended to fulfil.

Conical Projection is that in which the meridians and a selected parallel are conceived as being projected on to the surface of a cone which circumscribes the globe and is tangential to it at the selected parallel. The developed surface of the cone will reveal the selected or standard parallel at its true length and as the arc of a circle with its centre at the vertex. This arc will be divided into 360 equal arcs by the projected meridians which will appear as straight lines radiating from the vertex at equal angles. Other parallels will be arcs of circles concentric with the standard parallel and of some specifically selected radii. It is interesting to notice that the straight line which represents the Equator in cylindrical projection may be regarded as the arc of a circle of infinite radius, divided equally by the projected meridians which radiate from an infinitely remote vertex.

Conical Projection with two standard parallels, in which two parallels are selected for representation at their true lengths and at their true distance apart, is often called the '*Secant Conical*' on the assumption that the cone cuts the sphere in the two selected parallels. This is not the case, however, for a secant cone would show the distance between the parallels as a chord, whereas their true distance apart is the length of the arc of the meridian which they intersect. It is simply a 'projection' in which the meridians are shown as straight lines radiating from a point and in which the parallels, two of which are of their true length and distance apart, are shown as concentric arcs.

Zenithal Projection is that projection which is made on a plane conceived as being tangent to the globe. We have already conceived the cone as being increasingly flattened until the angle at the vertex reached the limit of 180°, when the cone had become a tangent plane touching the sphere at a pole. Projections on such a plane are known as *Polar Zenithal Projections*. The angle at the vertex of the developed



GNOMONIC

STEREOSCOPIC

ORTHOGRAPHIC

CLARKE'S MINIMUM ERROR

POLAR

NORMAL

OBLIQUE

ZENITHAL PROJECTION

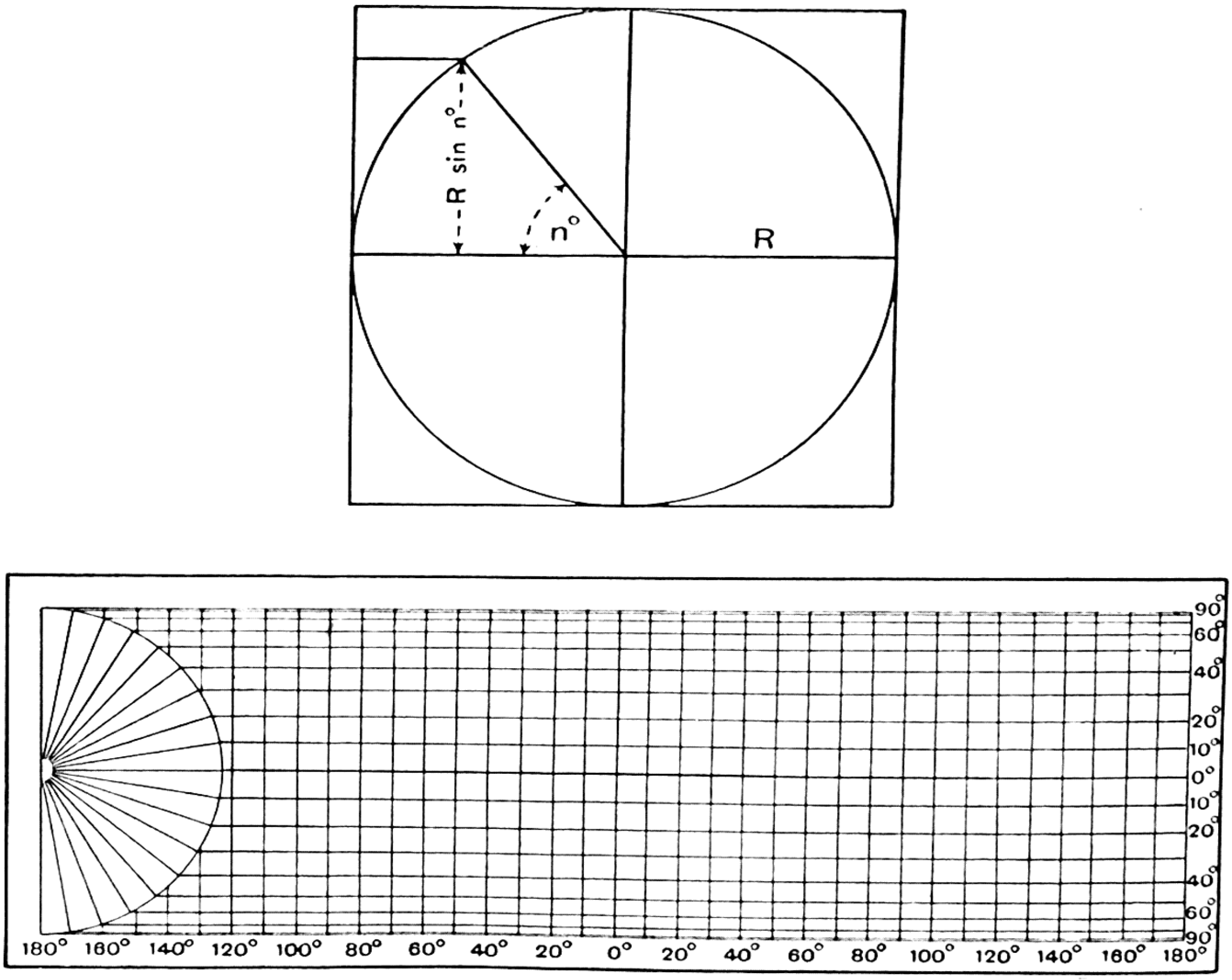
DIAGRAM 13

cone has become one of 360° and lies at the centre of the circle. From this point, which represents a Pole, the meridians radiate at their true angles. The parallels, instead of being arcs, will be complete circles having their centres at the Pole. The radii of these circles which represent the parallels may either be computed in accordance with some mathematical formula or they may be genuinely projected from some view point situated somewhere on the polar axis or the polar axis produced.

Obviously, the plane need not be tangent to the sphere only at a Pole. When it is tangent to the sphere at any point on the Equator the resultant projection is known as *Normal.* At any other point it is known as *Oblique.*

Whether the zenithal projection be polar, normal or oblique, however, the view point must be somewhere on that diameter of the globe which passes through the point at which the plane is tangent, or on that diameter produced. As all such lines pass through the centre of the globe it follows that the view point, in all forms of zenithal projection, lies on a line which is vertical at that point represented by the centre of the map. From the movability of the view point along this infinite line are derived four sub-divisions of the class of zenithal projections. *Gnomonic or Central Projection*, to which we referred in the previous section, is that in which the view point is at the centre of the sphere. *Stereographic Projection* is that in which the view point is at the opposite end of the diameter ; in Polar Zenithal this means obviously the other Pole. *Orthographic Projection* is that in which the view point is situated at infinity, so that the ‘ rays ’ are conceived as being perpendicular to the tangent plane. *Clarke's Minimum Error Perspective Projection* is that in which the view point is selected to give the greatest general accuracy for a specific map. The most useful position for the view point for this purpose is a point vertically distant from the point at which the plane is tangent by from 1·36 to 1·7 times the radius of the sphere.

Some of the more important modified or non-classifiable projections are discussed later.



CYLINDRICAL PROJECTION

DIAGRAM 14

II

CYLINDRICAL EQUAL AREA PROJECTION

IN **Cylindrical Equal Area Projection** the points on the sphere are projected on to the surface of a circumscribed cylinder by lines perpendicular to the sides of the cylinder. Assuming the cylinder to touch the sphere at the Equator, it follows that :

(1) *the Equator* will be a straight line, horizontally bisecting the map, and in length it will be 2π times the radius of the inscribed sphere ;

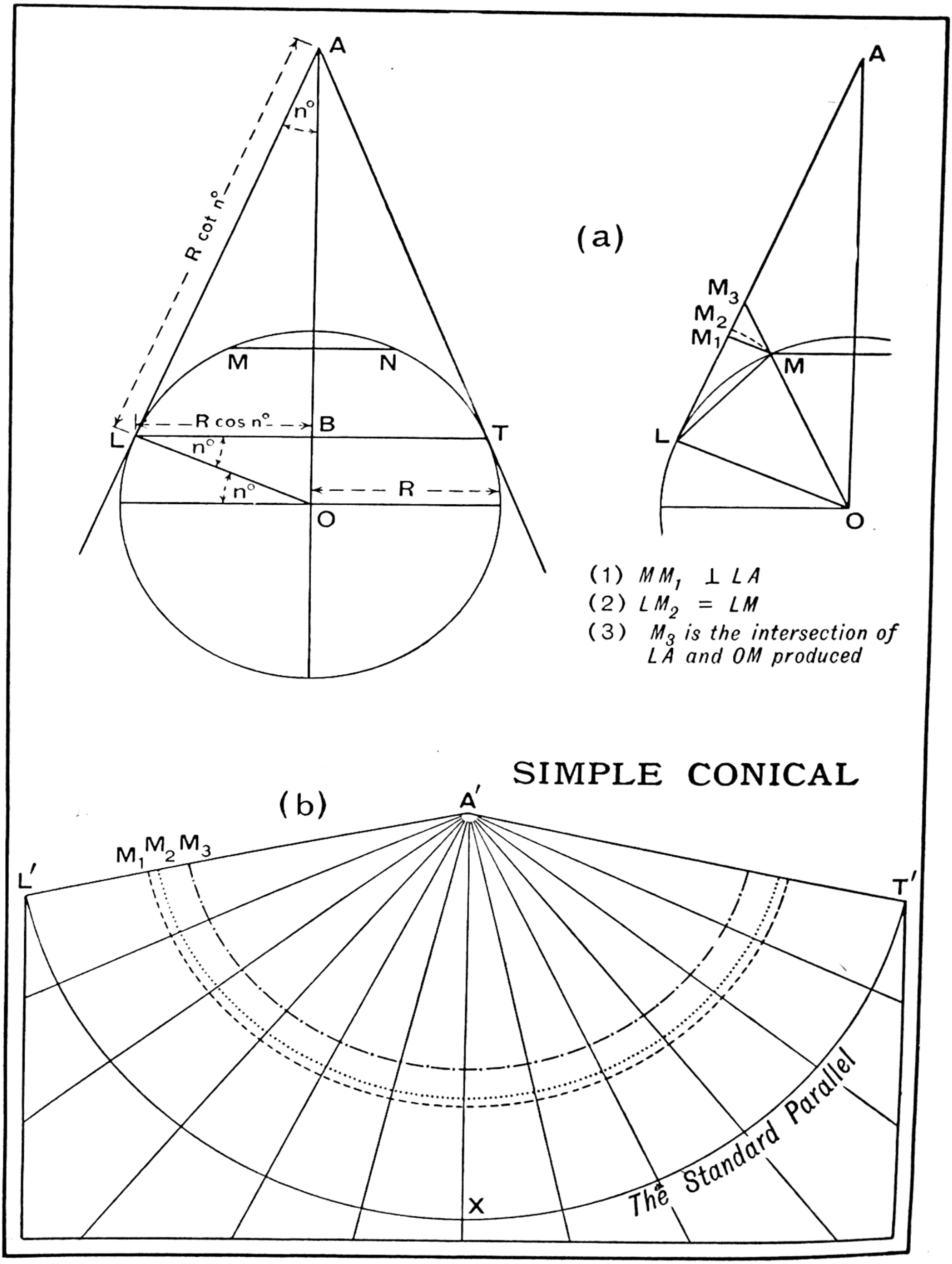
(2) *the meridians* will be parallel lines, perpendicular to the Equator, and dividing it into 360 equal parts ;

(3) the meridians will, therefore, cut off sections equal in area to the corresponding lunes of the globe, as the area of a sphere equals the area of the circumscribed cylinder ($4\pi R^2$) ;

(4) *the parallels* will be straight lines, parallel to the Equator, at distances from it proportional to their sines (see diagram) ;

(5) the parallels will cut off areas equal to the corresponding sections of the sphere. (The proof of this, though as old as Archimedes, is rather beyond the scope of the mathematics to which this book is restricted.)

This projection has little practical value owing to the excessive and increasing distortion of shape towards the Poles.



(1) $MM_1 \perp LA$

(2) $LM_2 = LM$

(3) M_3 is the intersection of LA and OM produced

SIMPLE CONICAL

DIAGRAM 15.—CONICAL PROJECTION WITH ONE STANDARD PARALLEL

III

CONICAL PROJECTIONS

(*a*) **Simple Conical, with one Standard Parallel.** The sectional diagram 15 (*a*) shows the tangent cone LAT touching the sphere of centre O at the parallel of latitude $n°$, represented by the line LT, intersecting AO in B.

The true length of the parallel is $2\pi LB$; i.e., $2\pi R \cos n°$ On the developed cone, diagram 15 (b) this circle will be projected as the arc $L'XT'$ at its true length of $2\pi R \cos n°$.

To construct the developed cone, i.e. the projection, diagram 15 (b).

(1) Angle $L'A'T'$ is $360 \sin n$ degrees (or $2\pi \sin n$ radians);

$$\text{because} \quad \frac{L'A'T'}{360} = \frac{\text{arc } L'XT'}{\text{circle of radius } L'A'} \text{ degrees}$$

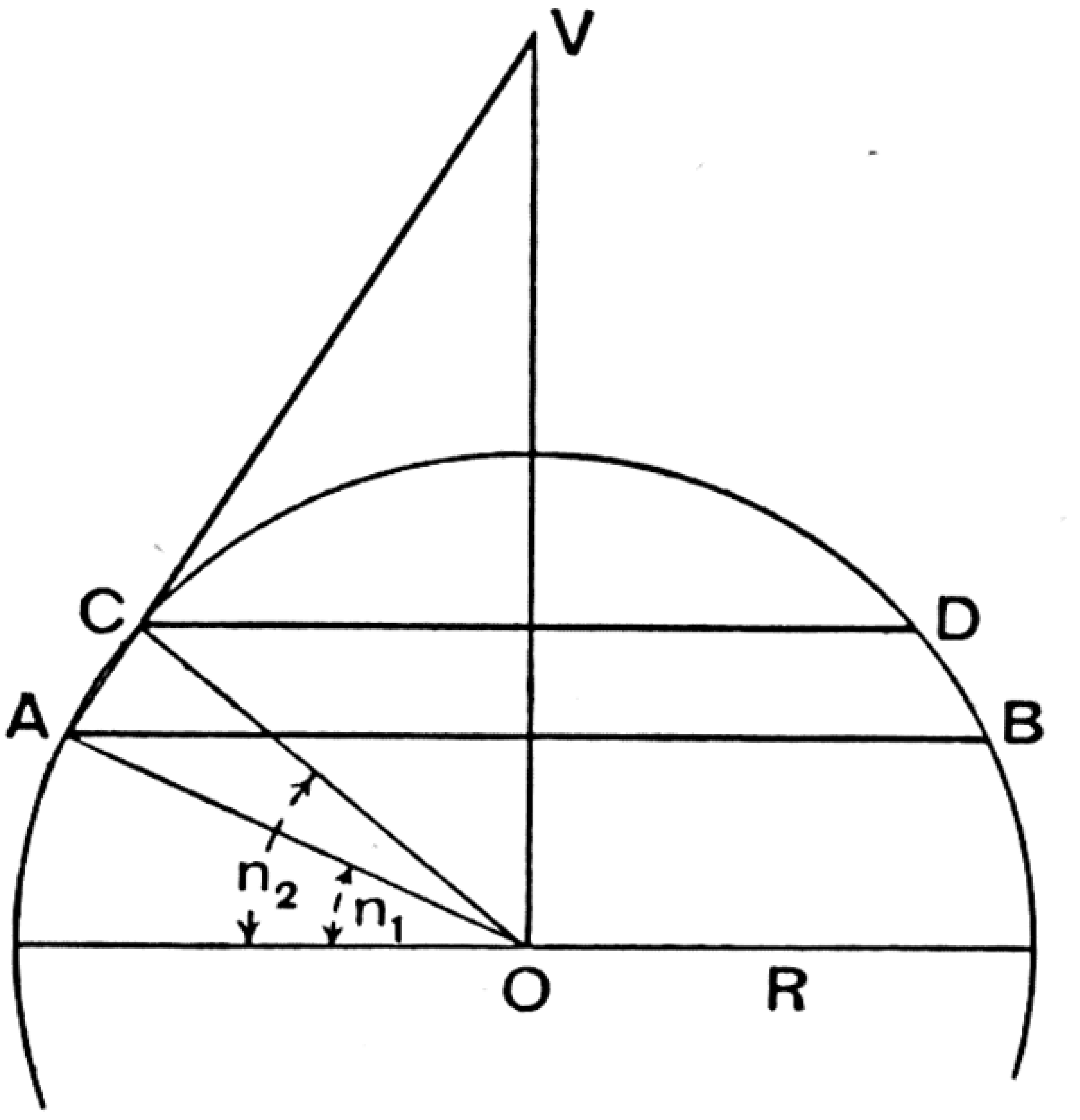
$$= \frac{2\pi R \cos n}{2\pi R \cot n} \text{ degrees (for } L'A' = LA)$$

$$= \sin n \text{ degrees; i.e. some fraction of a degree.}$$

This angle, $\frac{1}{360}$ of the angle at the vertex, is therefore the angle between two consecutive meridians in the projection. It is known as the constant of the cone and represented in this book by the symbol c.

(2) Having constructed angle $L'A'T'$, with centre A' and radius AL construct the arc $L'XT'$, which represents the standard parallel.

(3) Other parallels are shown by concentric arcs whose radii are selected to satisfy the specific purpose of the map. In diagram 15 (*a*) MN represents any other parallel. MM_1 is perpendicular to LA; LM_2 equals LM; OM is produced to cut LA in M_3. M_1, M_2 and M_3 are three possible representations of M, and arcs with respective radii AM_1, AM_2 and AM_3 are shown on the developed cone.



(a)

$$n_1^\circ = 25^\circ \qquad n_2^\circ = 40^\circ$$

$$\therefore c = \frac{\cos 25^\circ - \cos 40^\circ}{\frac{\pi}{180}\,(40^\circ - 25^\circ)}$$

$$= \underline{\underline{\cdot 536}}$$

(1) $\therefore \underline{\underline{\angle A'V'B' = 360^\circ\, c = 193^\circ}}$

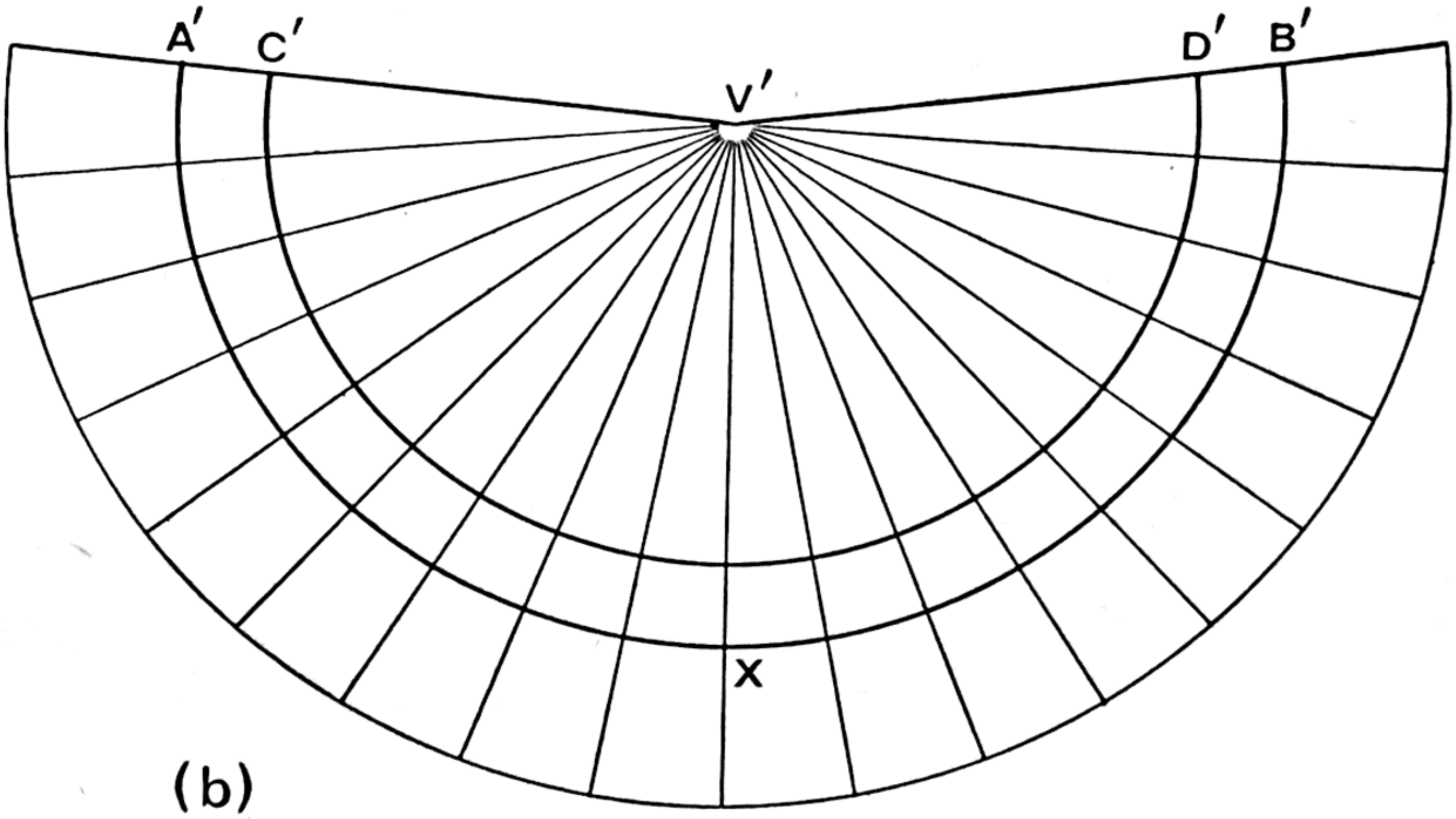
$R = 24$ mm. (selected)

$$\therefore r_1 = V'A' = \frac{24 \times \cos 25^\circ}{\cdot 536}$$

(2) $= \underline{\underline{40{\cdot}6 \text{ mm. (approx.)}}}$

$$A'C' = \text{arc } AC = \frac{15^\circ}{360^\circ} \times 2\pi \times 24$$

(3) $= \underline{\underline{6{\cdot}3 \text{ mm.}}}$



(b)

CONICAL PROJECTION
with Two Standard Parallels

DIAGRAM 16

(*b*) **Conical with two Standard Parallels** is a projection which represents two parallels at their true length, separated by their true distance, as concentric arcs from the centre of which the straight lines representing the meridians radiate. The choice of the errorless parallels is arbitrary ; in general they are a sixth of the total range of the latitudes.

Let the two selected parallels be of latitudes $n_1°$ and $n_2°$, and represented in diagram 16 (*a*) by AB and CD respectively. R is the radius of the sphere.

The true length of parallel $n_1°$ and ∴ of arc $A'B'$ is

$$2\pi R \cos n_1° ;$$

the true length of parallel $n_2°$ and ∴ of arc $C'D'$ is

$$2\pi R \cos n_2° ;$$

the true length of arc $A'C'$ is $\dfrac{2\pi R\,(n_2-n_1)°}{360}=\dfrac{\pi R\,(n_2-n_1)°}{180}$.

The problem is to find the angle $A'V'B'$ *and the radius* $V'A'$ (r_1).

$$(1)\quad \frac{A'V'B'}{360}=\frac{\text{arc } A'B'}{2\pi r_1}=\frac{2\pi R\cos n_1°}{2\pi r_1}=\frac{R\cos n_1°}{r_1}=c$$

(the constant of the cone).

$$(2)\qquad \frac{V'A'}{V'C'}=\frac{A'B'}{C'D'}=\frac{\cos n_1°}{\cos n_2°} ;$$

$$\therefore\ \frac{V'A'}{A'C'}=\frac{V'A'}{V'A'-V'C'}=\frac{\cos n_1°}{\cos n_1°-\cos n_2°} ;$$

$$\therefore\ r_1=V'A'=\frac{\text{arc } A'C'\cos n_1°}{\cos n_1°-\cos n_2°}=\frac{\pi R\,(n_2-n_1)\cos n_1°}{180\,(\cos n_1°-\cos n_2°)} ;$$

$$\text{from (1),}\qquad r_1=\frac{R\cos n_1°}{c} ;\ \therefore\ c=\frac{\cos n_1°-\cos n_2°}{\dfrac{\pi}{180}\ (n_2-n_1)°}.$$

Diagram 16 (*b*) shows a computation by this method, with the standard parallels selected at 25° N and 40° N. At the scale adopted, $R=24$ mm.

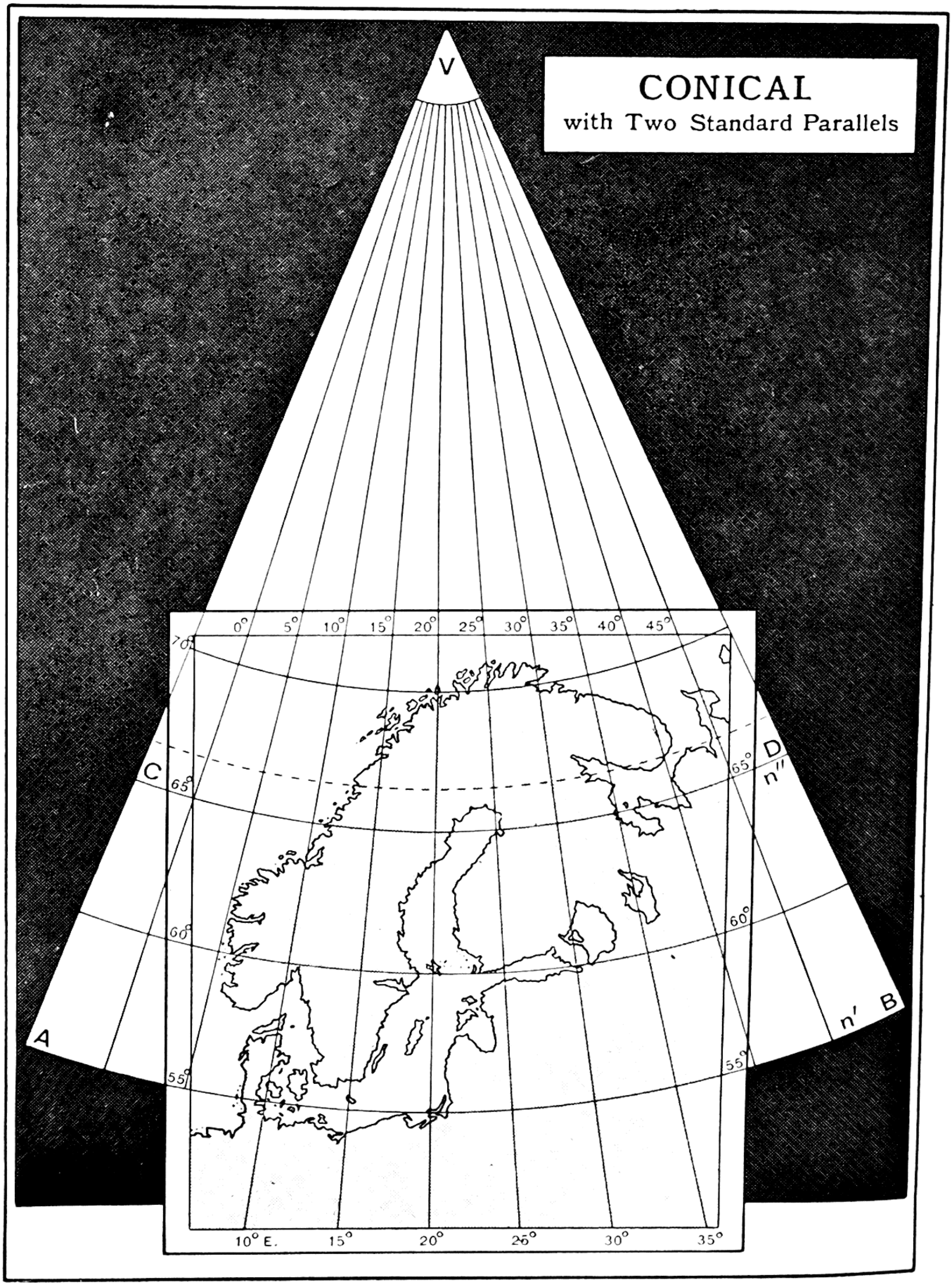


DIAGRAM 17.—USE OF CONICAL PROJECTION FOR MAP OF THE BALTIC LANDS

(*c*) **Map of Baltic Lands—An Example of Conical Projection with two Standard Parallels.** If every fifth parallel is to be shown, parallels of latitude 55° and 65° N. are the obvious selections.

(1) *To construct the meridians*

$$c = \frac{\cos 55^\circ - \cos 65^\circ}{\frac{\pi}{180} \times (65 - 55)} = \cdot 86\,;$$

∴ angle between each pair of selected fifth meridians will be 4·3°.

Meridian 20° E. is drawn as the central meridian and the other meridians are drawn from a convenient point V at angles of 4·3°.

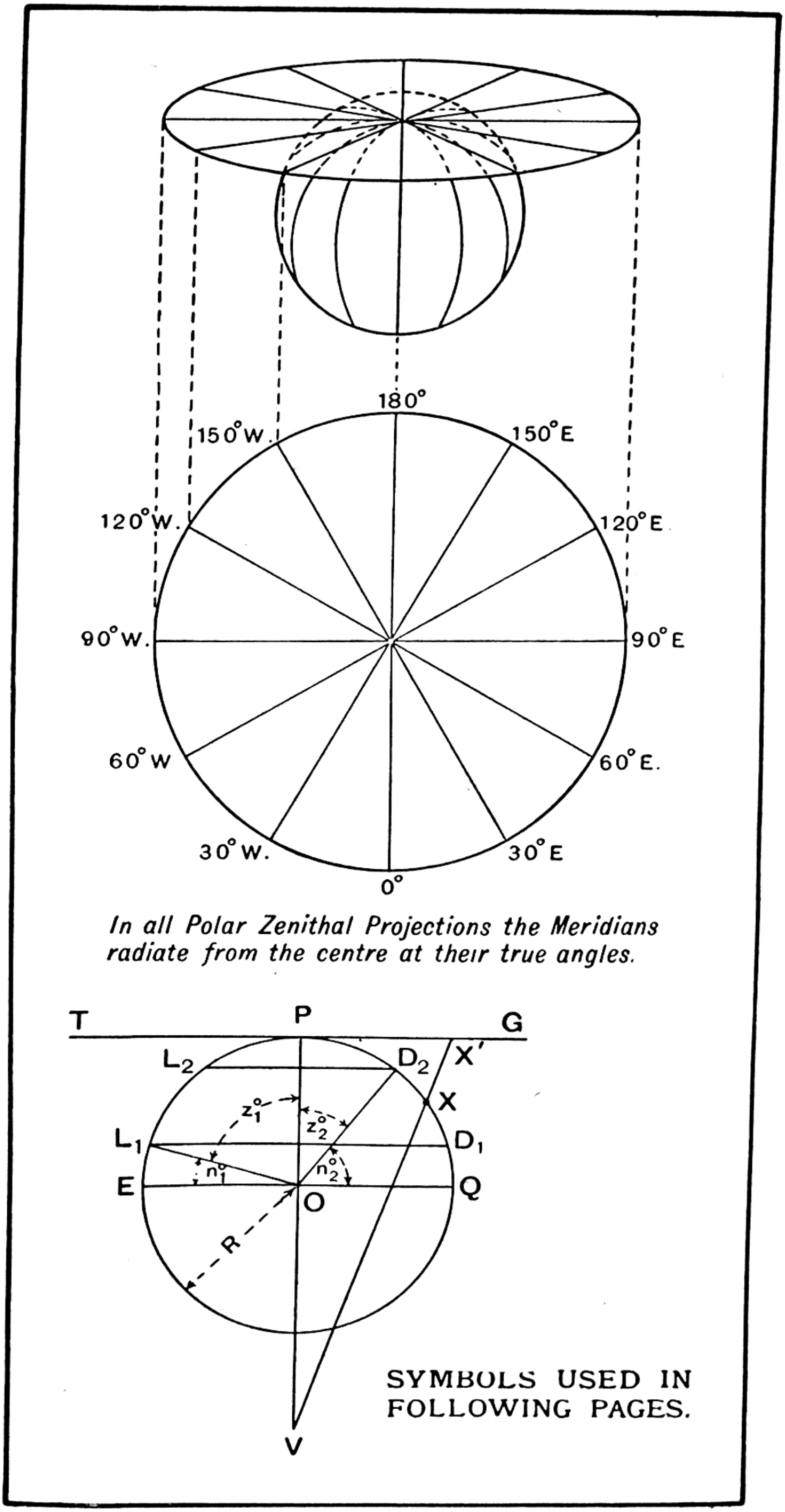
(2) *To construct the parallels*

In an atlas a frequent scale for this map is 1 : 10 million, when *R* equals 636·6 mm. Before reduction necessitated by the size of the page the map in the example was drawn to a scale of 1 : 20 million, when *R* equals 318·3 mm.

(*a*) $VA = r_1 = \dfrac{318\cdot 3 \times \cos 55^\circ}{\cdot 86} = \underline{212 \text{ mm.}}$

(*b*) $AC = \dfrac{\pi \times 318\cdot 3 \times 10}{180} = \underline{55\cdot 5 \text{ mm.}}$

After drawing the arcs *AB* and *CD*, construct other required parallels with proportionate radii.



In all Polar Zenithal Projections the Meridians radiate from the centre at their true angles.

SYMBOLS USED IN FOLLOWING PAGES.

Diagram 18.—General Characteristics of Polar Zenithal Projection

IV

POLAR ZENITHAL PROJECTIONS

GENERAL CHARACTERISTICS. As in all Polar Projections the centre of the map is one of the Poles, it follows that :

(1) the meridians will always be projected as straight lines radiating from the Pole at their true angles ;

(2) the parallels will be projected as concentric circles with the Pole as their centre ;

(3) the only possible difference, therefore, between one form of polar zenithal projection and another must lie in the relation of the radii of the concentric circles which represent the parallels.

In the following descriptions of polar zenithal projections it is assumed that the meridians are as in diagram 18, and the descriptions are, therefore, restricted to the projection of the parallels.

The symbols used are :

R = the Radius of the Sphere ;

V = the View Point, or centre of projection ;

O = the centre of the sphere ;

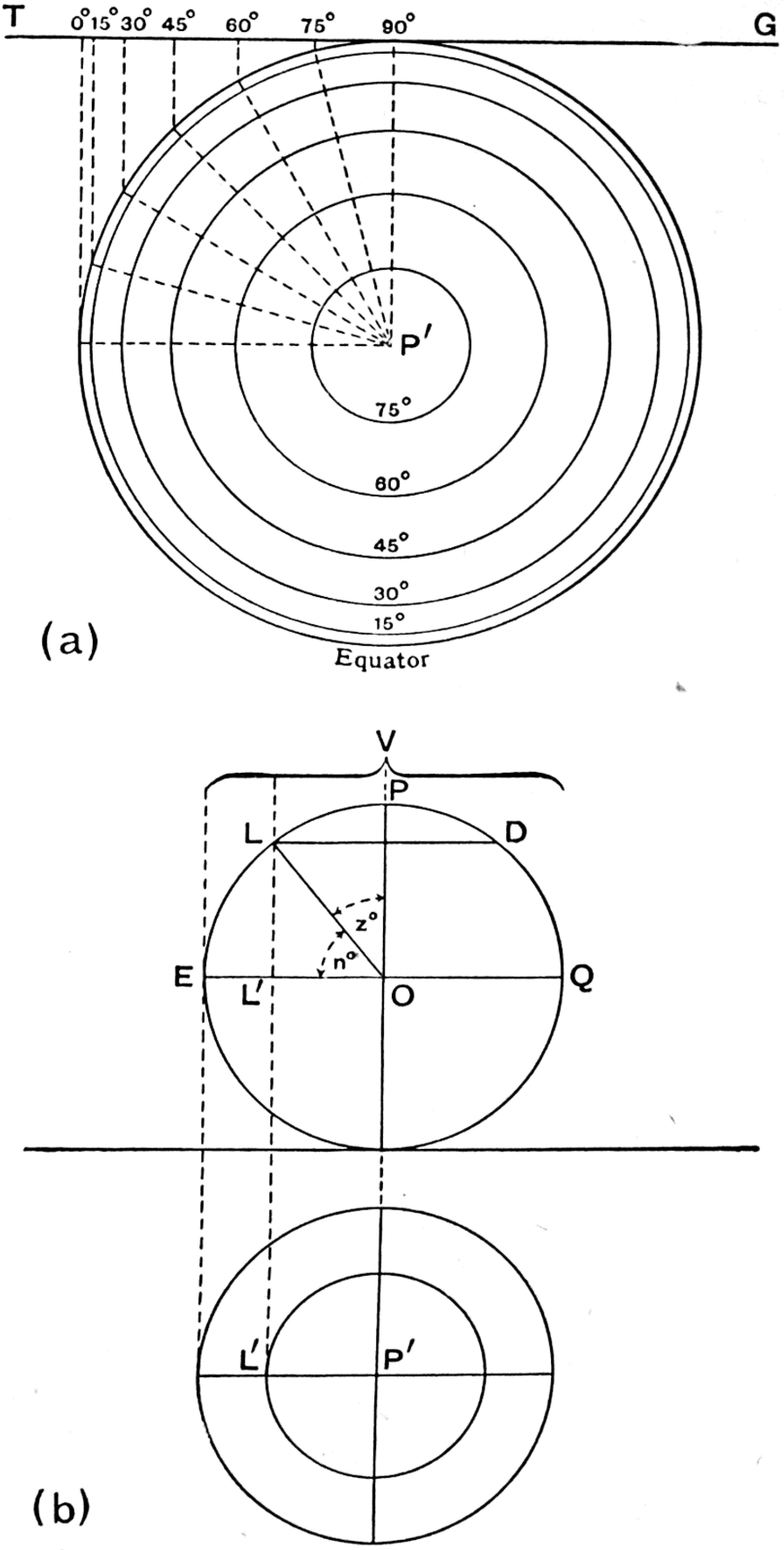
P = a Pole ;

TG = a tangent plane ;

EQ = the Equator ;

$L_1 D_1$, $L_2 D_2$ = parallels of latitude of n_1, n_2 degrees, and of co-latitude of z_1, z_2 degrees ;

X' = the projection of X.



ORTHOGRAPHIC POLAR PROJECTION

DIAGRAM 19

(*a*) **Orthographic Polar Zenithal** is a perspective view of the globe from an infinite distance directly over a Pole. Projection of all points, therefore, is made by lines passing through them perpendicularly to the plane.

The diagram 19 (*a*) shows the orthographic projection of every fifteenth degree of latitude.

The radii of the circles representing the parallels may be taken directly from *TG* or they may be mathematically computed.

In diagram 19 (*b*), angle *LOE* is $n°$, and angle *POL* is $(90-n)°$ or $z°$.

$$\therefore\ r = OL' = R \cos n° \ \text{ or } \ R \sin z°.$$

That is, *the radius of the circle representing any parallel in orthographic projection equals the sine of its co-latitude times the radius of the sphere.*

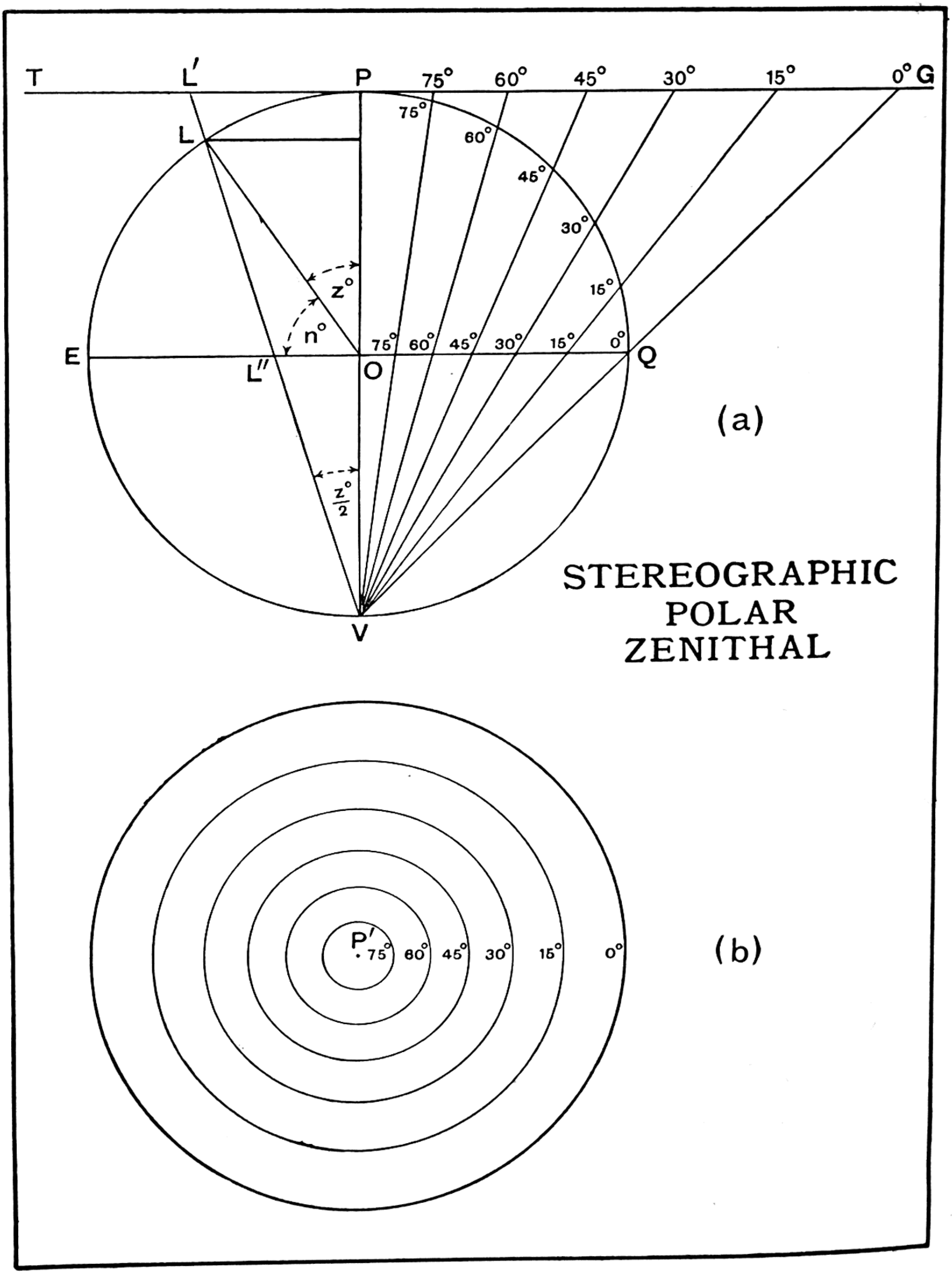

T
L′
P
75°
60°
45°
30°
15°
0° G
L
75°
60°
45°
30°
15°
z°
n°
E
L″
O
75° 60° 45° 30° 15° 0°
Q
(a)
z°/2
V
STEREOGRAPHIC
POLAR
ZENITHAL
P′
75° 60° 45° 30° 15° 0°
(b)

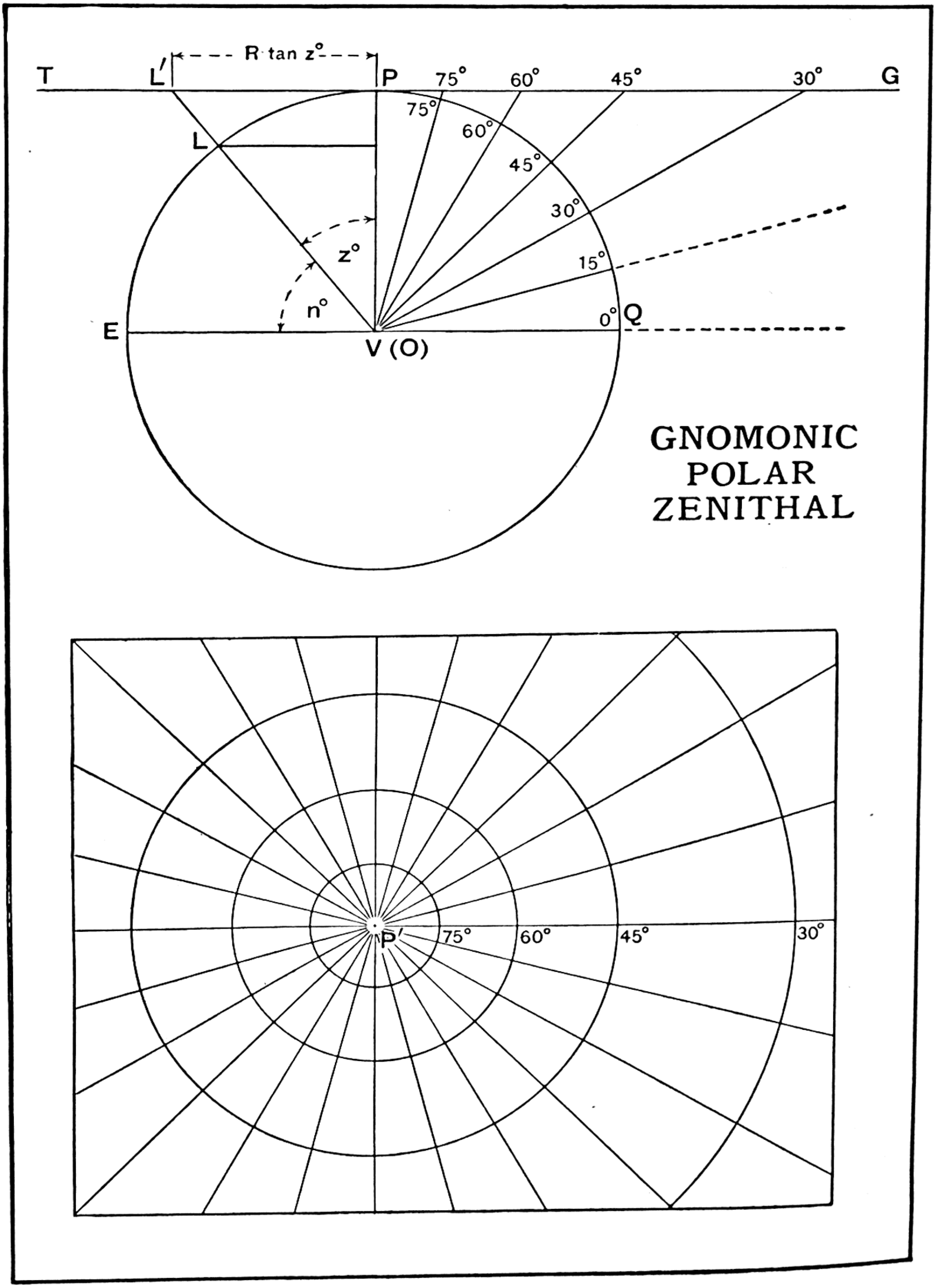
DIAGRAM 20

(*b*) **Stereographic Polar Zenithal** is a perspective view of the globe from the opposite Pole, as though the globe were a hollow sphere of glass.

In diagram 20 (*a*) the rays from V, which pass through the intersections of the parallels with the boundary meridian, intersect the tangent plane TG. It is geometrically obvious that they will intersect the Equatorial plane EQ, or any other parallel plane, proportionately. The projection, diagram 20 (*b*), was constructed from the scale of radii along EQ.

The radii may be mathematically computed; and from diagram 20 (*a*) it is clear that $OL'' = r = R \tan \frac{z}{2}$.

Therefore, in stereographic projection, the radius of any circle representing a parallel is the tangent of half its co-latitude times the radius of the sphere.



GNOMONIC POLAR ZENITHAL

Diagram 21

(*c*) **Gnomonic or Central Polar Zenithal** is a perspective projection with the view point situated at the centre of the sphere.

Both the graphic method of construction and the mathematical value of the radii ($r = R \tan z°$) are clear from diagram 21. It is also obvious that it is impossible to construct a gnomonic projection of a complete hemisphere ($\tan 90° = \infty$).

It is this projection which has the valuable characteristic to which previous reference has been made ; as the planes of all great circles pass through the centre of the sphere, in this case the view point, all great circles will appear as straight lines and will, therefore, be represented by straight lines on the map. *Gnomonic Projection, therefore, shows the shortest spherical distance between any two points by a straight line.* It is therefore a useful projection for navigation charts.

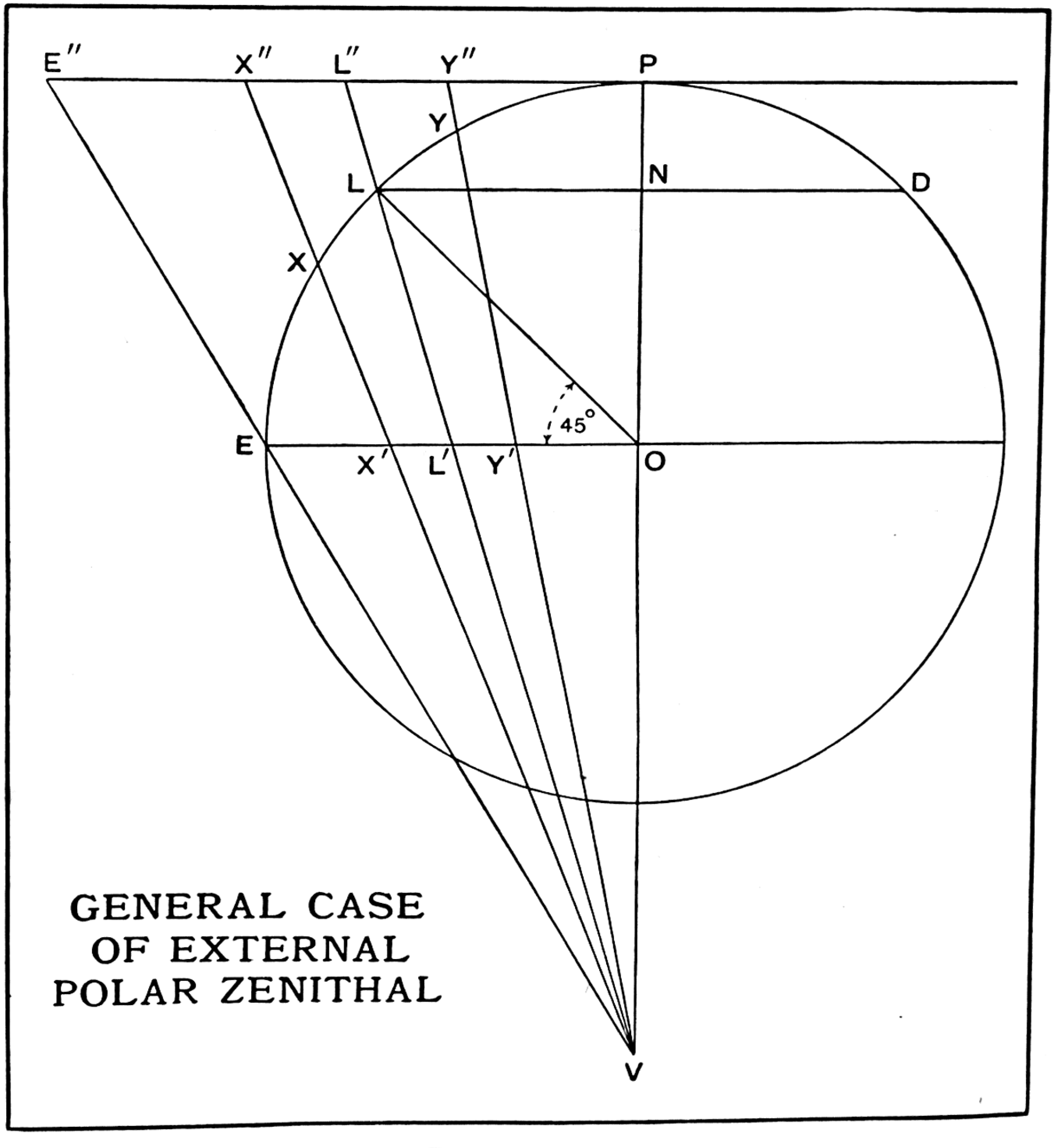

E″
X″
L″
Y″
P
Y
L
N
D
X
45°
E
X′
L′
Y′
O
GENERAL CASE
OF EXTERNAL
POLAR ZENITHAL
V

Diagram 22

(*d*) **The General Case of External Polar Zenithal.** By selecting a view point at a convenient finite distance external to the sphere the relation between the radii of the circles representing the parallels may be modified in relation to a desired end, as in Clarke's Minimum Error Projection.

In La Hire's Polar Zenithal the view point is selected so that the circles representing the parallels are practically equidistant.

In diagram 22, L bisects the quadrant ELP and $\therefore n=45°$.

L' is the mid-point of EO, and $L'L$ produced cuts the tangent plane in L''. V is the intersection of PO and $L''L'$ produced. VE produced cuts the tangent plane in E''. L'' will then bisect $E''P$.

V is then the external view point which projects the parallel which is midway between the Equator and the Pole by a circle midway between the Equator and the Pole on the map.

Will other points on the quadrant, similarly projected, divide EO or TP in the same ratio as they divide the quadrant ELP? X and Y, for example, trisect the quadrant ELP; do X'' and Y'' trisect $E''P$?

They appear to do so and the possibility is sufficiently interesting to justify its examination :

(1) *To calculate the length OV.*

Let $OV=b$, and let LD intersect PV in N. R is regarded as unity.

$$\frac{L'O}{OV}=\frac{LN}{NV}=\frac{LN}{OV+NO}=\frac{\cos 45^\circ}{b+\sin 45^\circ}, \quad \therefore L'O=\frac{b\cos 45^\circ}{b+\sin 45^\circ}.$$

But $$L'O=R/2=\tfrac{1}{2}, \quad \therefore b=\frac{\sqrt{2}}{2(\sqrt{2}-1)}=1+\frac{\sqrt{2}}{2}$$

or $$b=1{\cdot}707 \text{ times the radius.}$$

(2) $X'O=\dfrac{b\cos 30^\circ}{b+\sin 30^\circ}=\cdot 670$, which is slightly greater than 2/3;

$Y'O=\dfrac{b\cos 60^\circ}{b+\sin 60^\circ}=\cdot 332$, which is slightly less than 1/3.

There is, therefore, slight distortion towards E and O, or E'' and P.

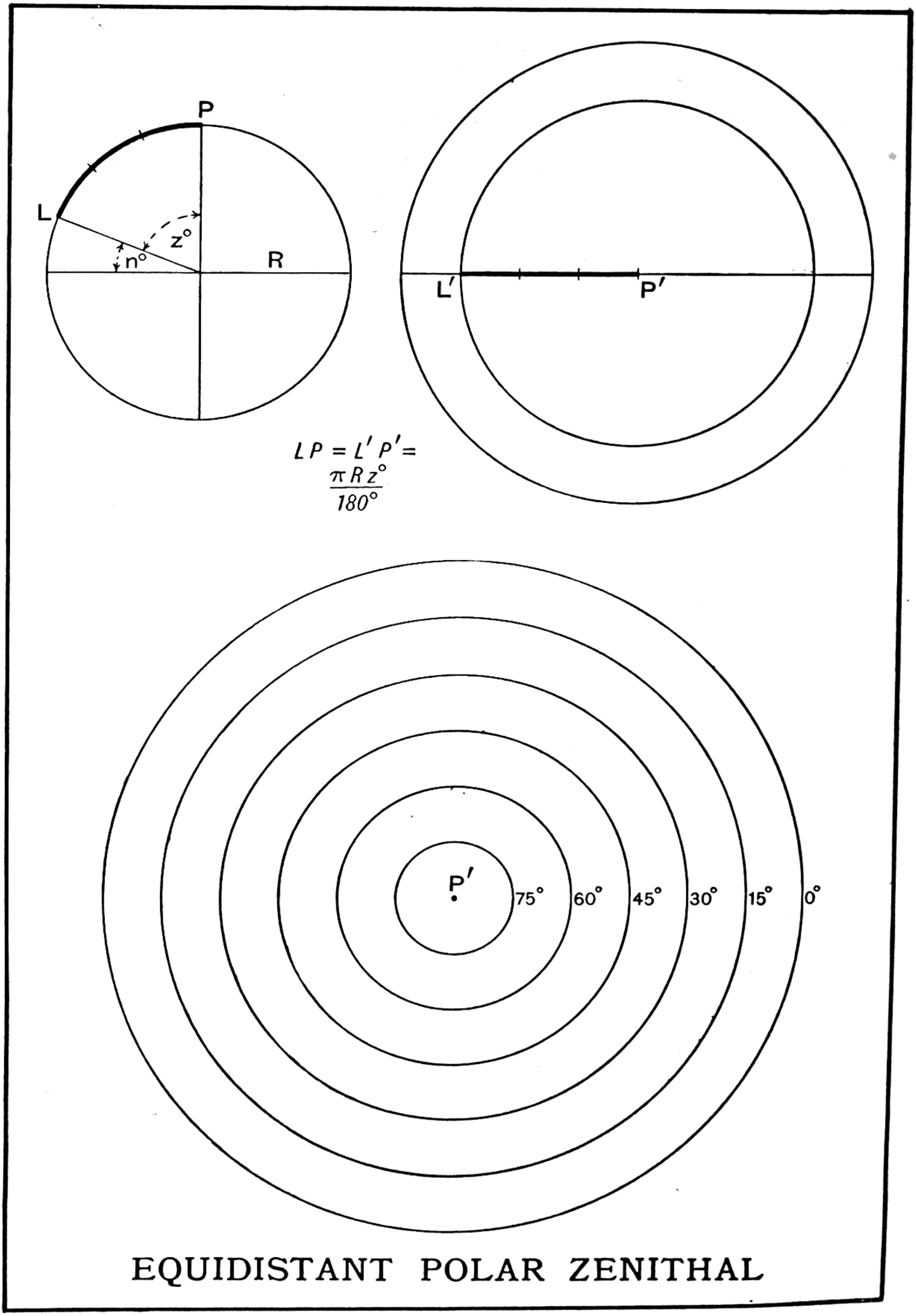
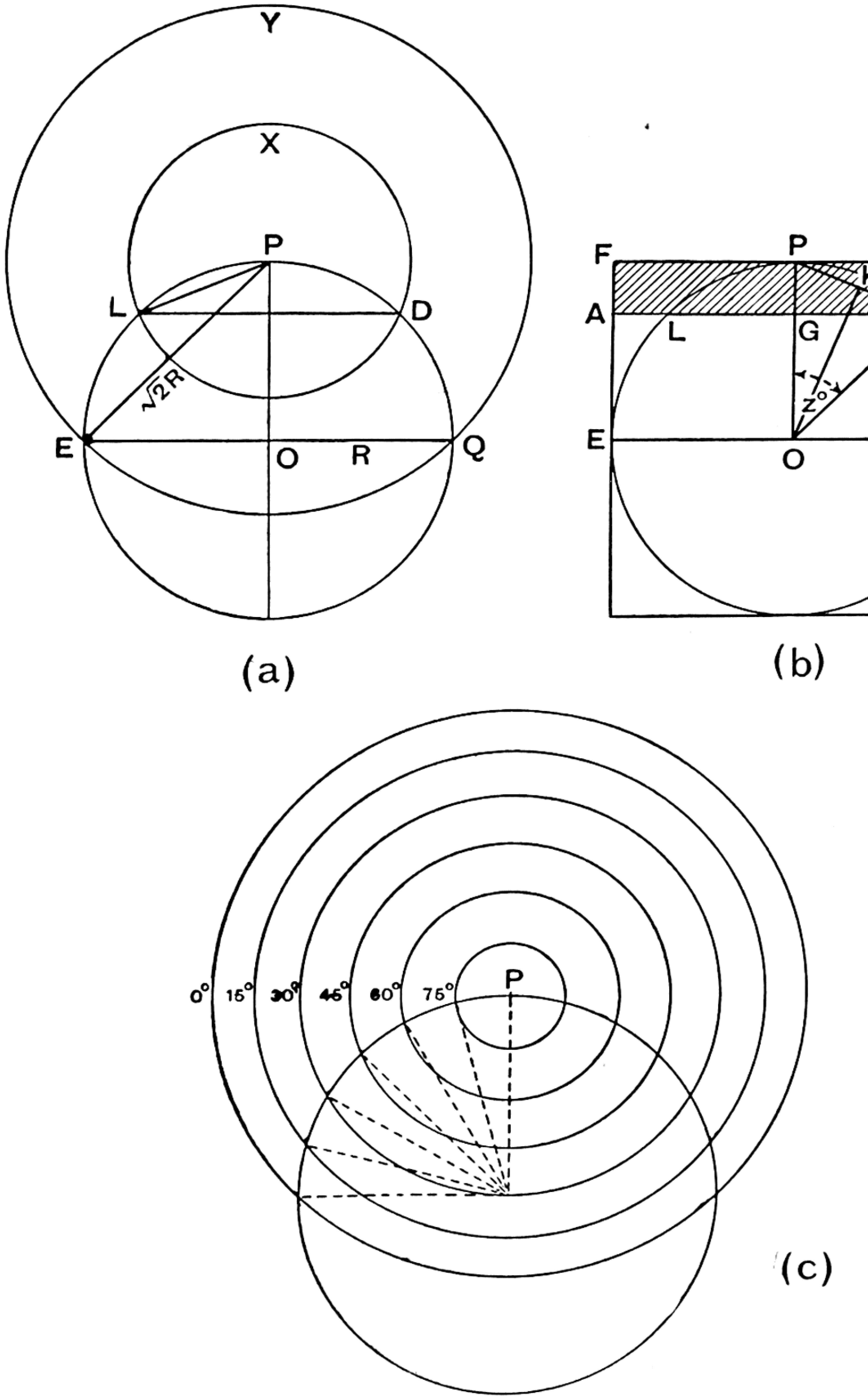

P
L
z°
n°
R
L′
P′
LP = L′P′ = πRz°/180°
P′
75°
60°
45°
30°
15°
0°
EQUIDISTANT POLAR ZENITHAL

DIAGRAM 23

(*e*) **Polar Zenithal Equidistant.** It follows from the preceding analysis that no position of *V* gives absolutely accurately an *equidistant projection* of the parallels of latitude. If equal divisions are required, therefore, the radii of the concentric circles will simply be chosen directly to divide the radius of the polar map into the requisite number of equal parts. Such a projection is an *arbitrary projection*, not a perspective projection.

While it is obvious that in this projection all distances along the meridians, that is, all distances measured from the Pole, will be true distances—distances correctly represented,—the parallels will be in proportion too long.

The projection, however, combines the virtues of being both an equidistant and an azimuthal projection.



LAMBERT'S POLAR ZENITHAL
(Equal Area)

Diagram 24

(*f*) **Lambert's Polar Zenithal Equal Area Projection.** In this 'projection' the radii of the circles representing the parallels equal the chords of their co-latitudes. (See diagram 24 (*a*)). *EP*, the radius of the circle which is the projection of the Equator and the boundary of the map, equals $R\sqrt{2}$, that is, $\sqrt{2}$ times the radius of the sphere projected.

Given that the area of a section of the circumscribed cylinder, measured from the pole for any angular distance z, equals the surface area of the 'cap' of the sphere enclosed in the section of the cylinder—the shaded portion in diagram 24 (*b*), the truth of which we have previously assumed :

To prove that this projection represents areas equally :

Area of section of circumscribed cylinder measured from the pole of any angular distance z, the shaded portion of diagram 24 (*b*), equals

$$2\pi R \times BC = 2\pi R(OP - OG) = 2\pi R(R - R\cos z)$$
$$= 2\pi R^2(1 - \cos z).$$

But the chord of any subtended angle $z = 2R \sin z/2$.

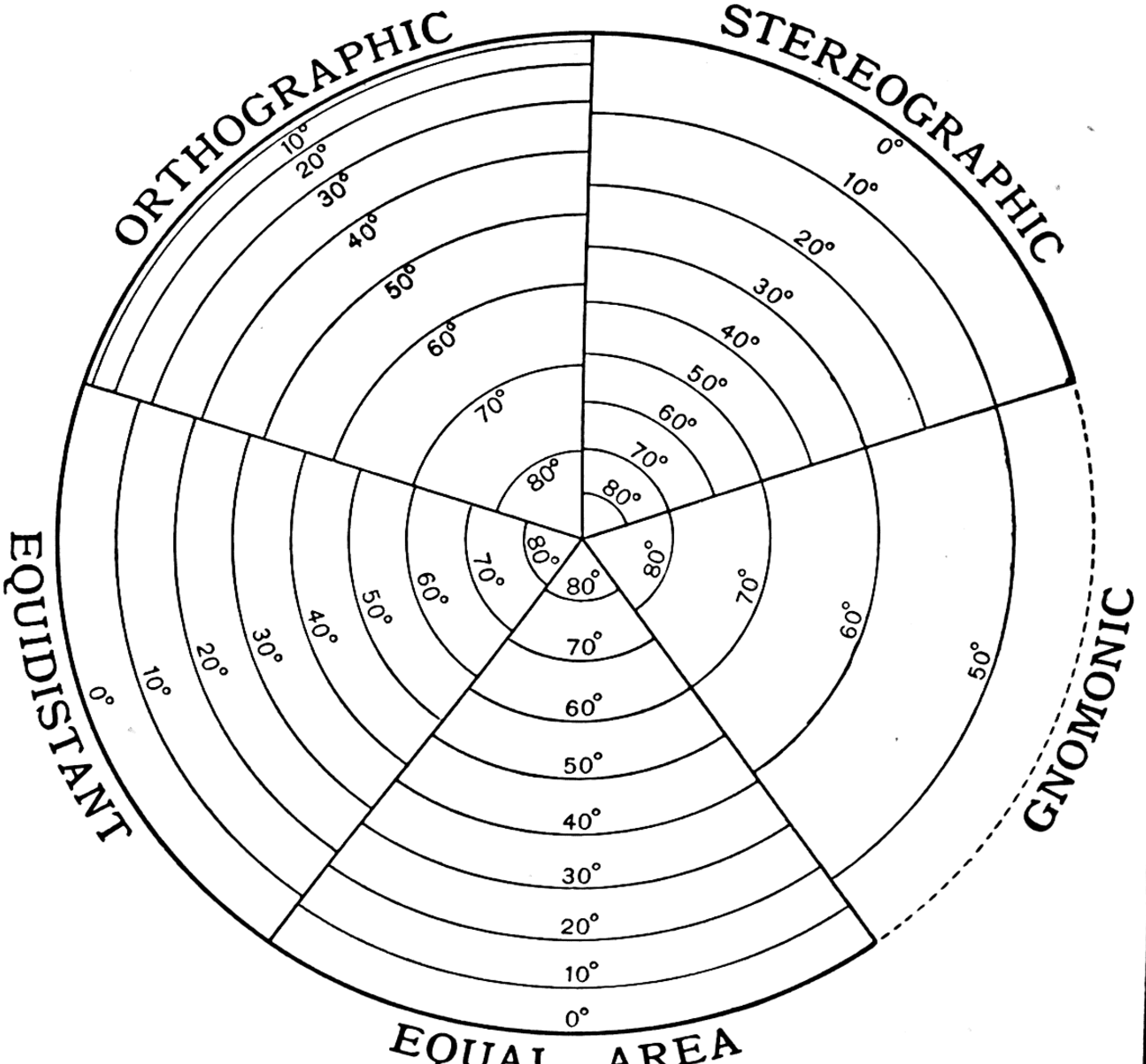
$\therefore$ area of circle with this radius $= \pi(2R\sin z/2)^2$

$$= 2\pi R^2 \times 2\sin^2 z/2$$
$$= 2\pi R^2(1 - \cos z)$$

$\therefore$ *Area of any circle of radius equal to the chord of an angle* z *equals the area of that section of the circumscribed cylinder which encloses the chord and, therefore, from the above premise, to the surface area of that section of the sphere enclosed.*

It follows that the area of the sphere bounded by two parallels, the difference in area between two such 'caps', will be correctly represented between the two concentric circles *whose radii equal twice the sines of half their co-latitudes multiplied by* $\sqrt{2}$ *times* the radius of the sphere ; i.e. $2\sin z/2 \,.\, \sqrt{2}R$.

It follows too that a scale of $\sin z/2$ will give the ratio in which the radii are related.



To illustrate the differences between the scales which represent the ratios of the radii of the circular projections of the parallels in polar zenithal projections.

DIAGRAM 25.—POLAR ZENITHAL PROJECTIONS CONTRASTED (*g*) **Summary and Comparison of Scales.** In the following table, the radius of the circular map, *r*, is regarded as unity.

Parallel n	80°	70°	60°	50°	40°	30°	20°	10°
Co-latitude z	10°	20°	30°	40°	50	60°	70°	80°
ORTHOGRAPHIC $r \sin z$	·174	·342	·500	·643	·766	·866	·940	·985
STEREOGRAPHIC $r \tan z/2$	·087	·176	·268	·364	·466	·577	·700	·839
GNOMONIC $r \tan z$	·176	·364	·577	·839	1·192	1·732	2·747	5·671
ARBITRARY $r\dfrac{90-n}{90}$	·111	·222	·333	·444	·555	·667	·778	·889
EQUAL AREA $r\sqrt{2R}\sin z/2$	·123	·246	·366	·484	·598	·707	·811	·909

26.—SUMMARY OF SCALES

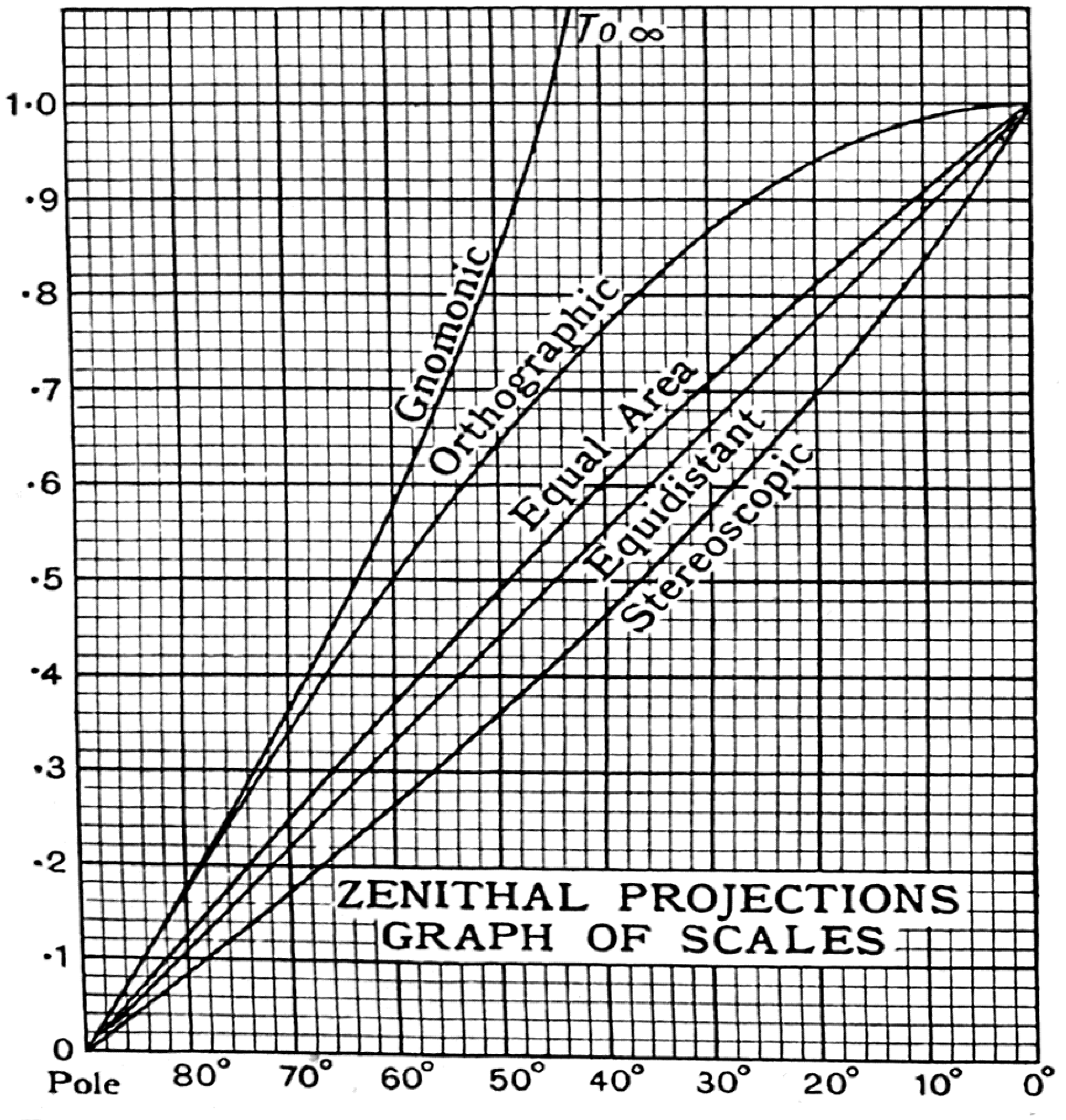
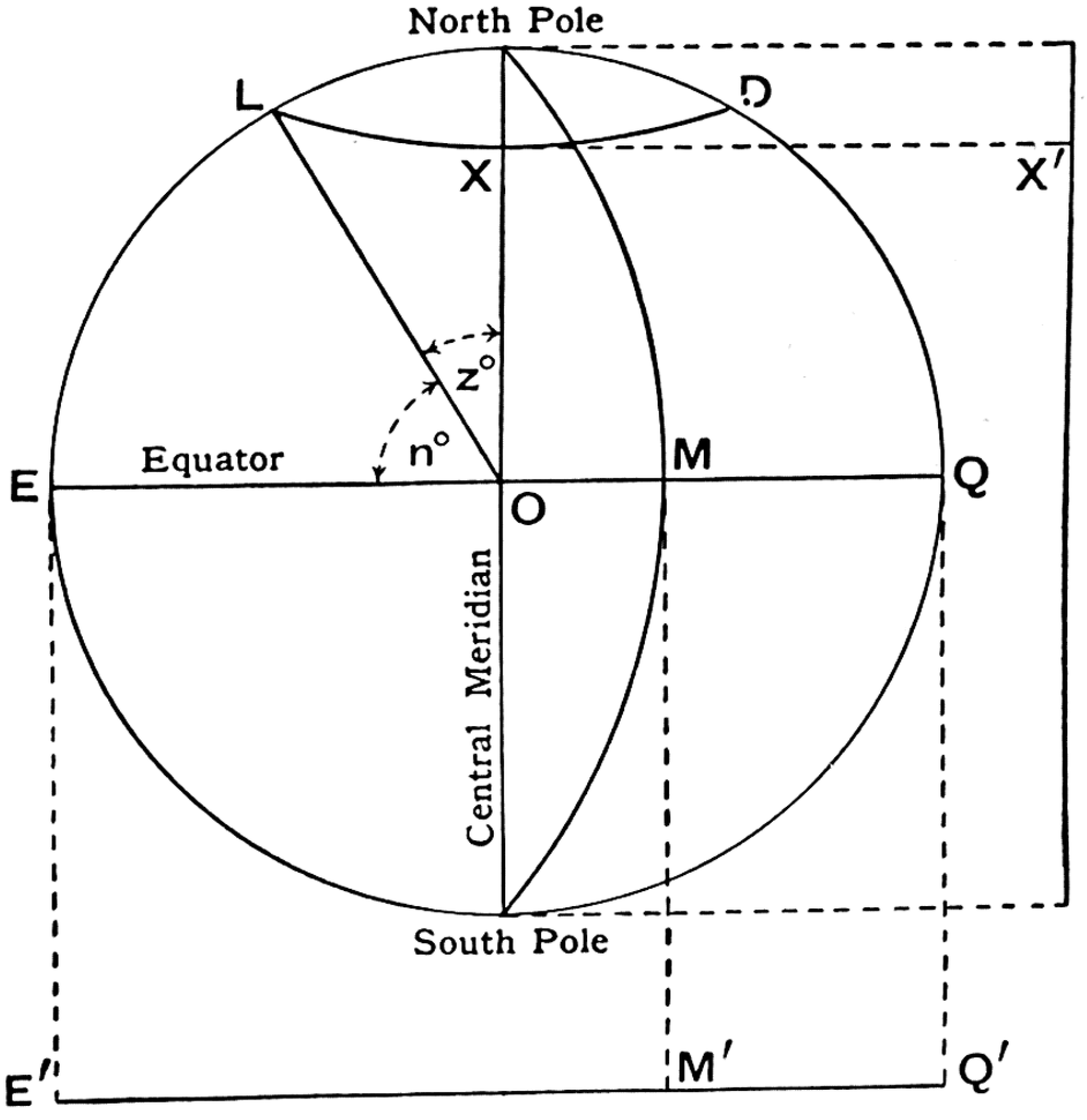


DIAGRAM 27.—GRAPHIC COMPARISON OF SCALES



Central Meridian to be divided in accordance with appropriate scale.

Equator to be divided in accordance with appropriate scale.

NORMAL ZENITHAL (GENERAL CHARACTERISTICS)

DIAGRAM 28

V

NORMAL ZENITHAL PROJECTION, WHEN THE PLANE IS TANGENT TO THE SPHERE AT A POINT ON THE EQUATOR

General Characteristics.

(1) As the plane of projection is tangential to and perpendicular to the plane of the Equator, *the Equator will be represented as a straight line which divides the map horizontally and equally.*

(2) A meridian, passing through the centre of projection, will be represented as a straight line which divides the map vertically and equally.

(3) All other meridians, as they pass through the Poles, will be represented by curved lines.

(4) As the points where the meridians intersect the Equator bear a similar relation to the point of vision in this projection as do the intersections of the parallels with the boundary meridian in polar projection, *the meridians may be marked off along the projected Equator according to the appropriate scale.* (See diagram 28.)

(5) As all parallels are divided equally by the meridians, *the meridians will divide each parallel in the projection by the same scale as they do the Equator.*

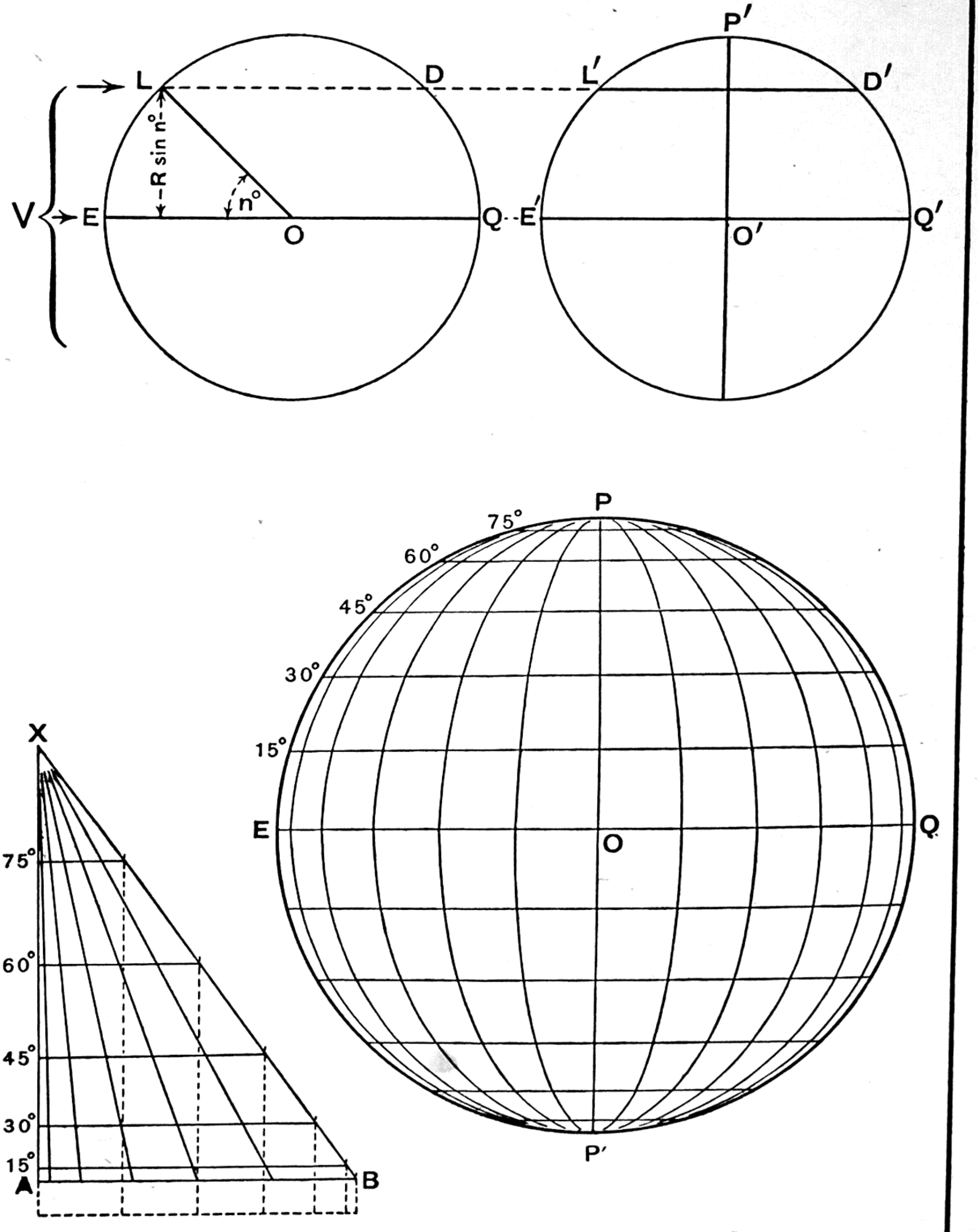
(6) As on the sphere the parallels divide every meridian into equal arcs, and *as the circle bounding the map will itself be a meridian, it will be divided equally by the parallels in any projection.*

(7) The parallels will divide the central meridian according to the appropriate scale.

From the above, it follows that it is possible to mark immediately, in any Normal Zenithal Projection, the following positions:

The Equator, the Poles, a central meridian, the boundary meridian, the intersections of any parallels with the boundary meridian, the intersections of any parallels with the central meridian, and the intersections of the meridians with the Equator. In the case of the last two possibilities the appropriate trigonometrical scale may be used.

This provides three points through which the curve representing any meridian or any parallel must pass.



ORTHOGRAPHIC NORMAL ZENITHAL

Diagram 29

(*a*) **Orthographic Normal Zenithal.** In Orthographic Normal Projection, as *V* is in the plane of the Equator, but infinitely remote, it follows that :

(1) *the parallels*, by virtue of their being in parallel planes, will be projected as straight lines parallel to the Equator ;

(2) *the meridians* will be constructed by dividing the Equator and the parallels according to the table of sines and by drawing curves through the points of intersection thus obtained.

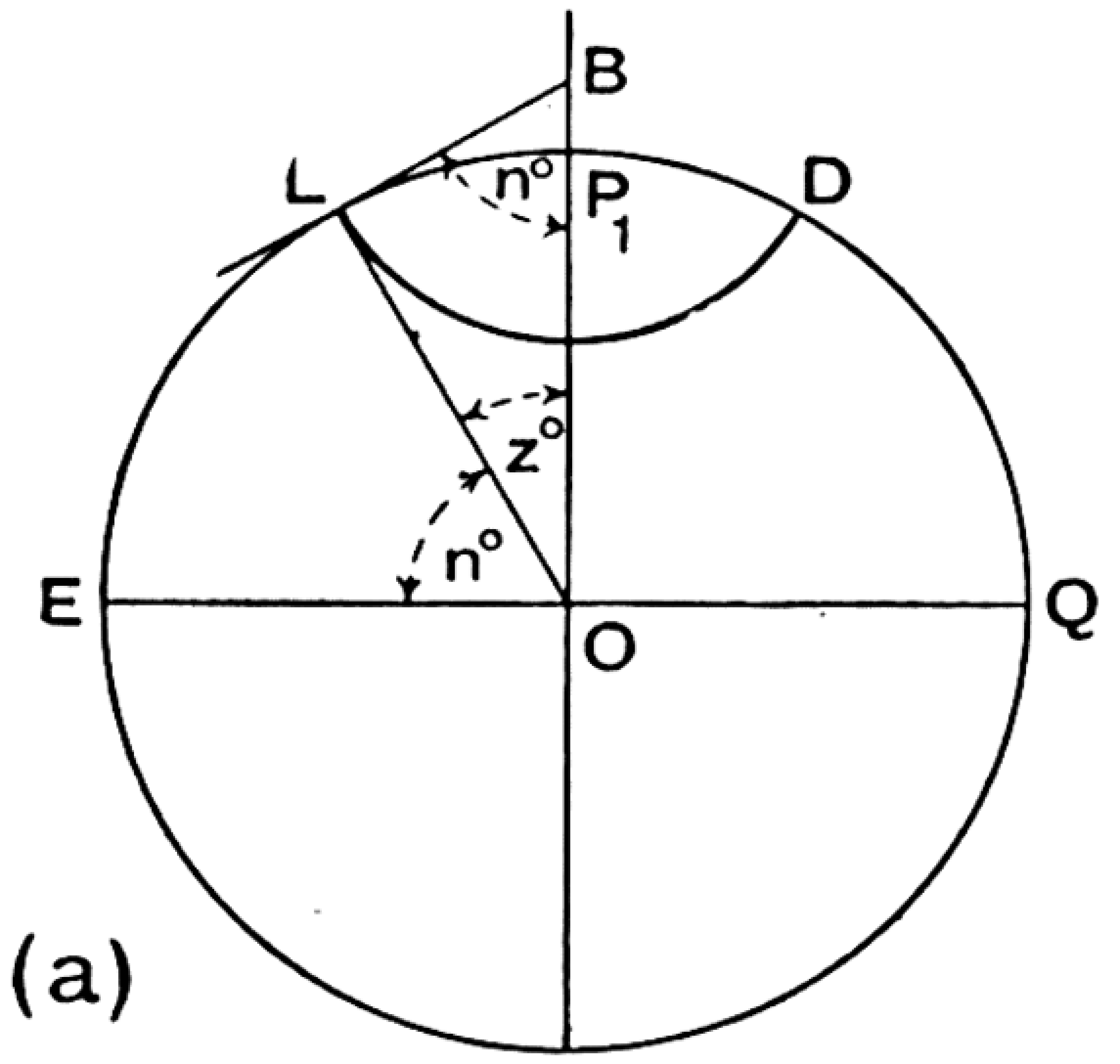
Graphic Method (See diagram 29).

Using a protractor, mark off equal divisions on semicircle *PEP′* and through these points draw the parallels.

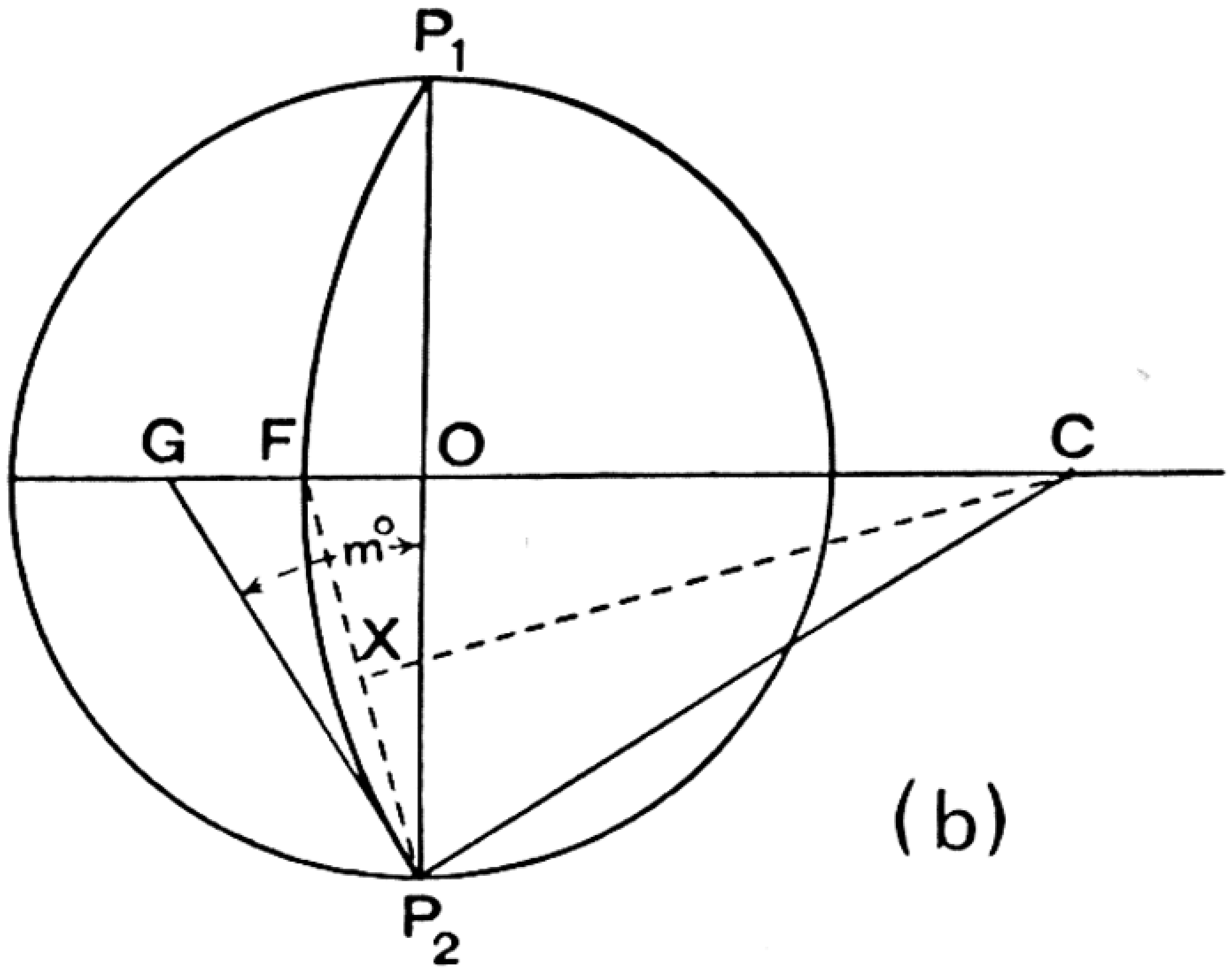
Mark along *EO*, *OQ* and *AB* (which equals *EO*) distances equal to the divisions of *PO*.

XA, of any convenient length, is perpendicular to *AB*.

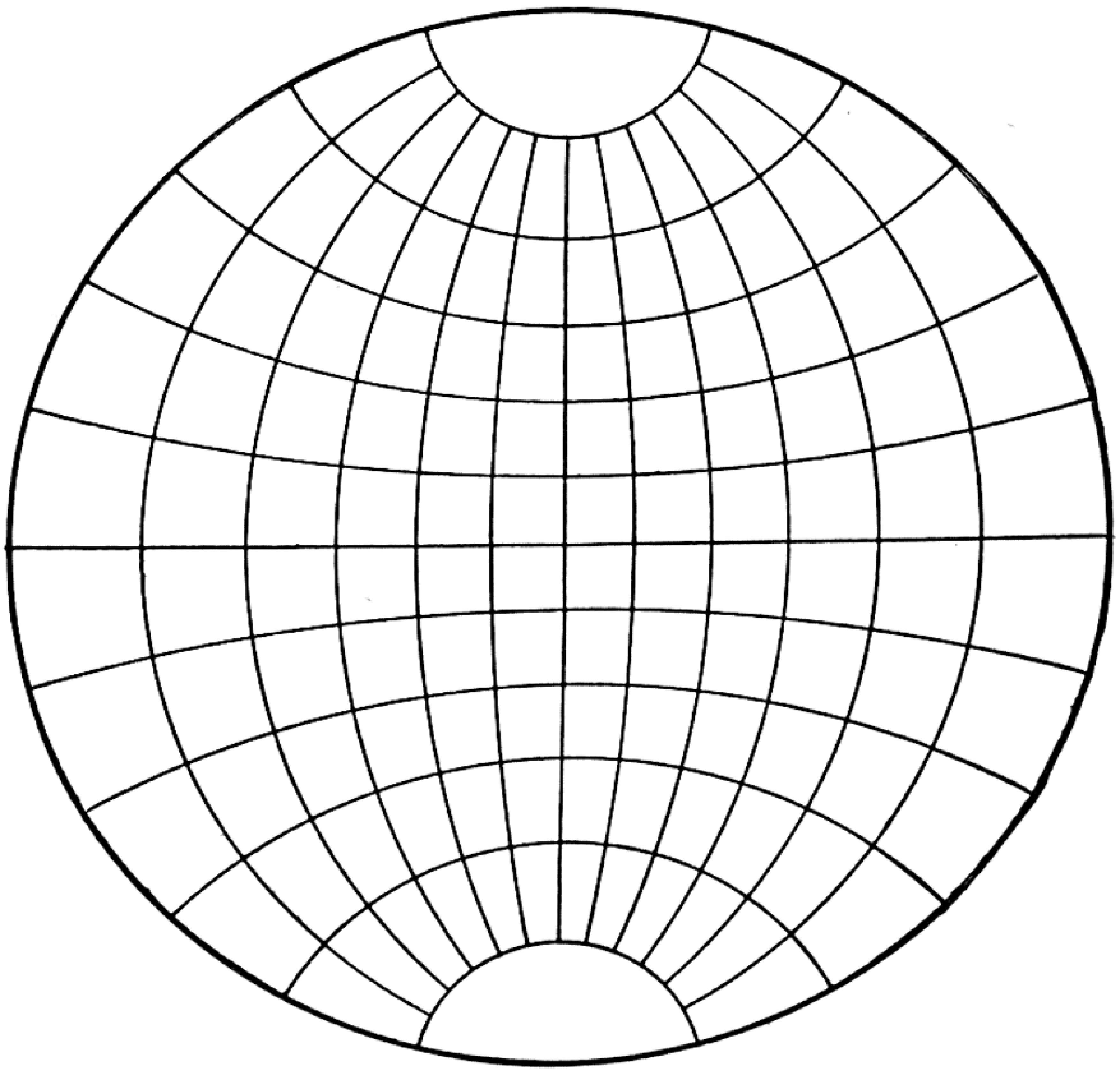
Join *XB*. Draw lines parallel to *AB* intersecting *XA* and *XB* and equal respectively to half the lengths of the parallels. Lines drawn from *X* to the points marked along *AB* will divide the lines representing the half parallels in the same proportion and the divisions can then be transferred to the map. Through the points thus found the curves representing the meridians can then be drawn.



B is centre of arc LD
the projection of parallel n°
$BO = R \operatorname{cosec} n°$
BL, the radius, $= R \cot n°$



O is centre of arc $P_1 F P_2$*,*
representing meridian of m°
$CP_2 = R \operatorname{cosec} m°$
$CO = R \cot m°$
(As FO is $R \tan \frac{m°}{2}$ *by construction,*
$\angle OP_2F = XCP_2 = XCF = FP_2G = \frac{m°}{2}$
$\therefore \angle OP_2G = m°$*)*



STEREOGRAPHIC NORMAL ZENITHAL

DIAGRAM 30

(*b*) **Stereographic Normal Zenithal.** All stereographic projections have two important characteristics :

(1) *All angles on the sphere are reproduced as equal angles in the projection.*

(2) *All circles on the sphere are reproduced by circles in the projection* (except those which pass through the plane of vision and which, appearing as straight lines, may be regarded as circles of infinite radius). For a proof of these propositions see Appendix A.

If the preliminary work common to all this class of projections is done, the marking of equal divisions on the circumference and the marking of the Equator and central meridian according, in this case, to a scale or $r \tan z/2$, there will already be three points through which every meridian and every parallel will have to pass. It follows from (2) above that *the only further work necessary is to draw arcs of circles through these points.* Centres will obviously lie on the line of the Equator produced or on that of the central meridian produced.

It follows from (1) above that the arcs representing the parallels will cut the boundary meridian at right angles, as they do on the sphere, and, therefore, centres of the arcs will lie on tangents to the boundary circle at these points of intersection.

The radius of each such circle, therefore, will be cot $n°$ *times the radius of the circular map and the distance of its centre from O will be* $r \operatorname{cosec} n°$.

Similarly, the centres of the meridians will lie on normals (lines perpendicular to the tangents) to the projections of the meridians at the Poles. As by construction the meridians intersect the Equator according to a scale of $\tan m/2$, it is geometrically obvious from diagram 30 (*b*) that the angle between the tangent to meridian of longitude $m°$ and the central meridian is m degrees. *Therefore, the radius of each meridional arc will be* $r \operatorname{cosec} m°$ *and the distance of its centre from O will be* $r \cot m°$.

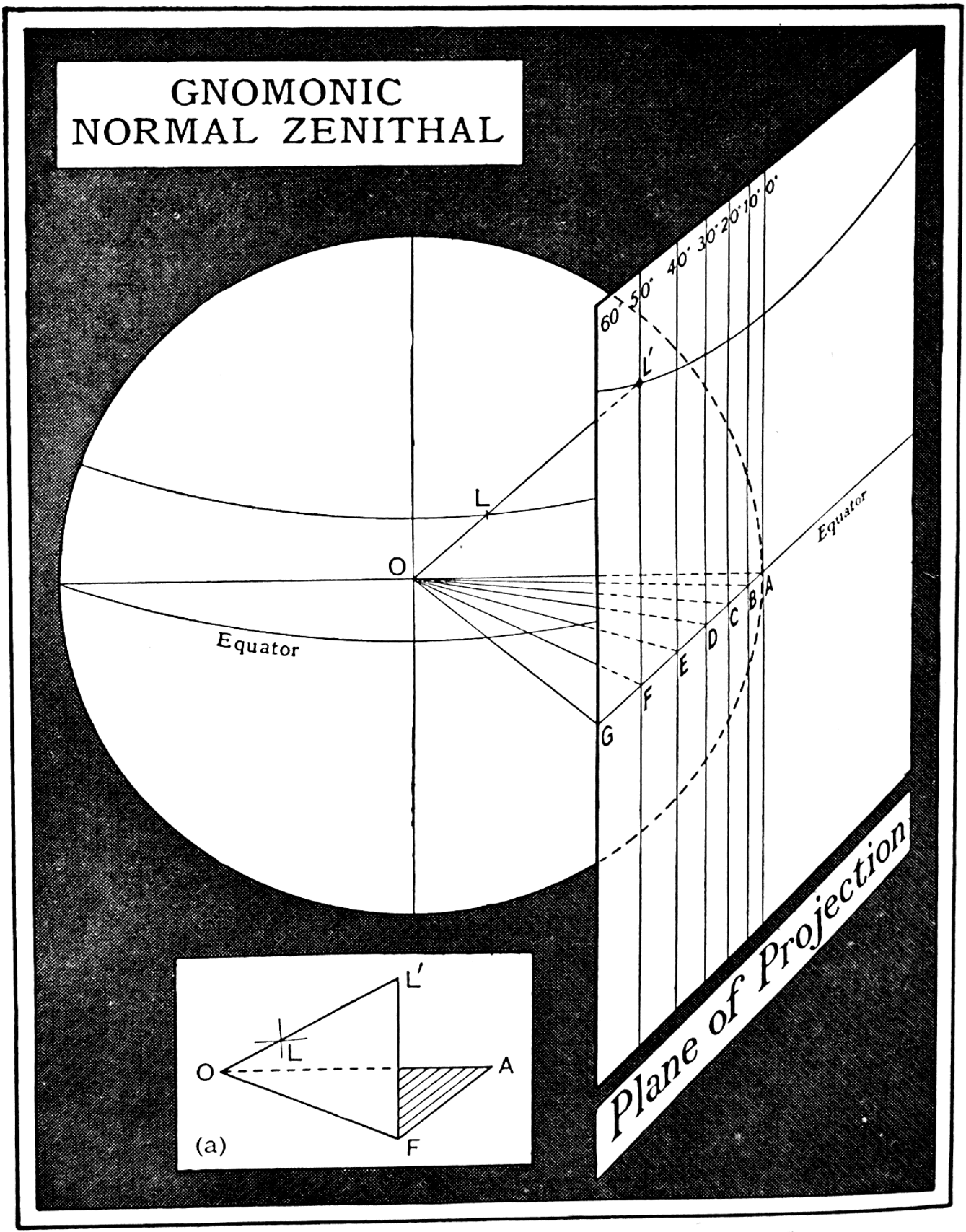


DIAGRAM 31.—PRINCIPLE OF GNOMONIC NORMAL ZENITHAL

(*c*) **Gnomonic Normal Zenithal.** Certain characteristics of all Gnomoic Projections have already been noticed, namely, that it is impossible to project a whole hemisphere, as the Poles are infinitely remote, and that all great circles are represented by straight lines. It follows, therefore, that in Gnomonic Normal Projection, *all meridians are straight lines, perpendicular to the central line which is the projection of the Equator.* This will be clear from diagram 31, as too will be the fact that the meridians intersect the Equator in accordance with the scale already found to be characteristic of Gnomonic Projections, namely, a scale of trigonometrical tangents.

After the Equator and meridians have been constructed the next stage is to plot the points of the intersections of the parallels.

In diagram 31 (*a*) let L represent the intersection of parallel $n°$ and meridian 50° and let L' be the projection of the point L. Let A be the central point of the projection, the intersection of meridian 0° and the Equator. O is the centre of the sphere and is therefore the view point. The radius OA is regarded as unity.

As AF is a tangent the angle OAF is a right angle ; angle AOF is 50°, $\therefore$ OF equals sec 50°.

In triangle $L'OF$ the angle $L'FO$ is a right angle and angle $L'OF$ is $n°$.

$\therefore$ $L'F$, the required distance, equals $OF \tan n° = \tan n° \sec 50°$.

The intersections of the parallels, therefore, with any meridian are plotted in accordance with a scale of tangents multiplied by secant longitude.

A simple graphic method of constructing this projection is described overleaf.

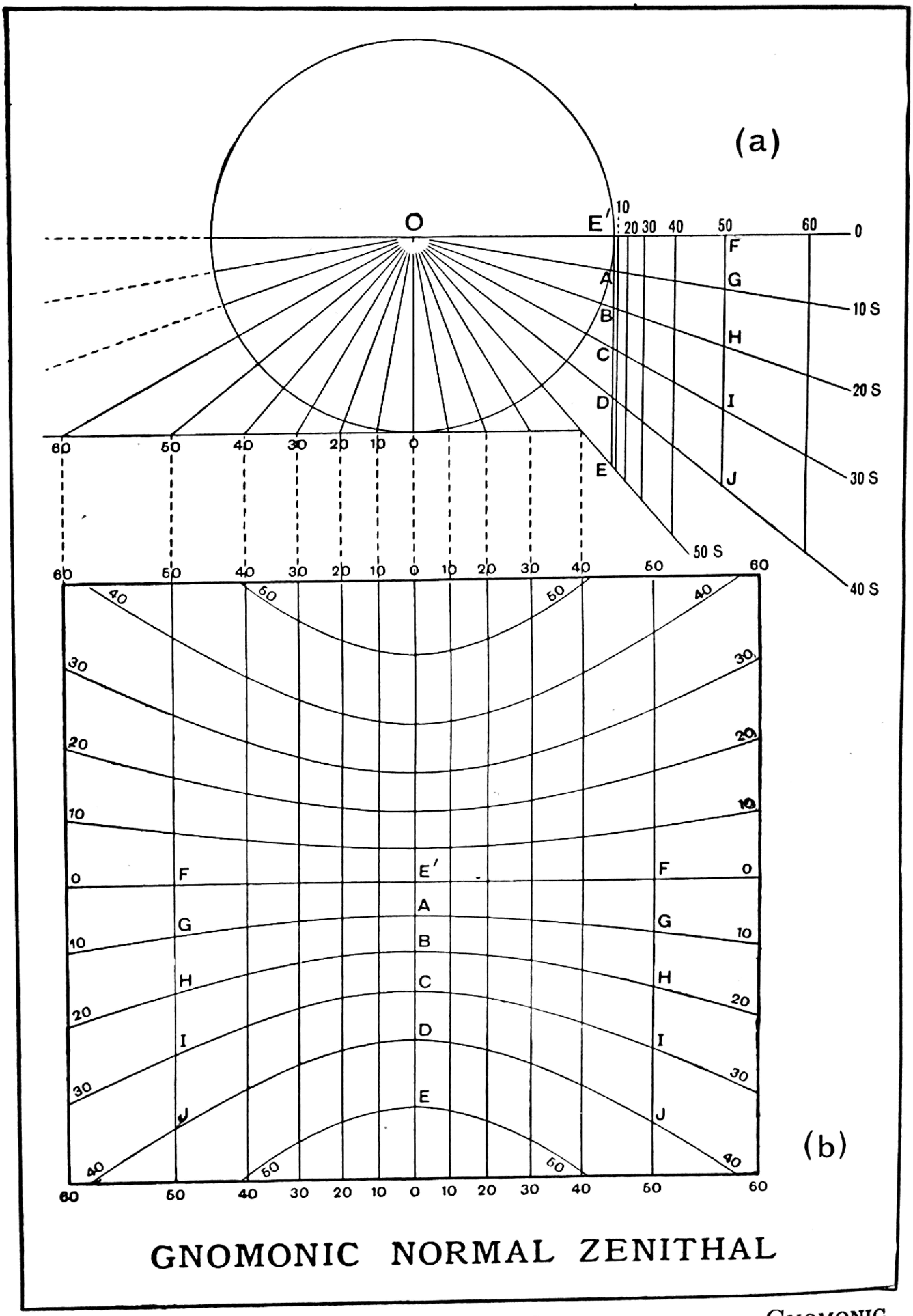


Diagram 32.—Graphic Method of Construction of Gnomonic Normal Zenithal

Graphic Method of Constructing Gnomonic Normal Projection

To construct the meridians it is clear from diagram 32 (*a*) that all that is necessary is to draw radii at the desired angles and produce them to intersect a tangent which represents the projection of the Equator. The intervals cut off will obviously be in the relation of tan longitude (tan $m°$).

To plot the parallels. We have already found that the distance along any meridian $m°$ where any parallel $n°$ intersects it, measured from the Equator, is tan $n°$ sec $m°$.

In diagram 32 (*a*) OE', a horizontal radius, is produced, and $E'E$ is a tangent to the circle at E'.

The radii are produced to cut this tangent in A, B, C, D, etc.

Then OA, OB, OC, etc. equal the secants of the longitudes, if the radius is regarded as unity.

These lengths are marked along OE' produced and perpendiculars are drawn through these points.

The intervals cut off along these perpendiculars by the radii produced give the required distances.

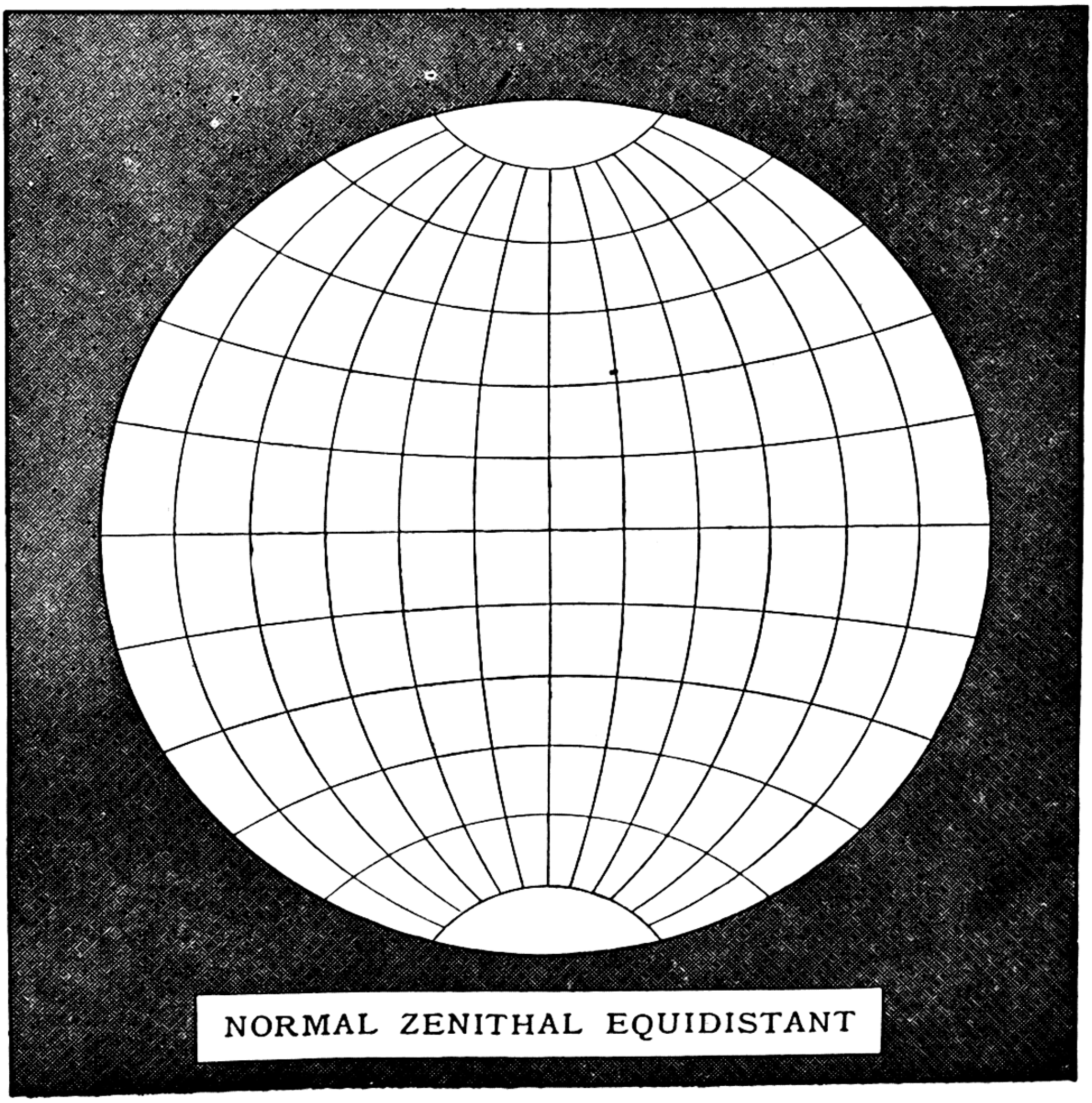
For example, OF by construction is sec 50°.

FG, therefore, is OF tan 10° = tan 10 sec 50.°

Similarly, FH is tan 20 sec 50° and so on.

These distances can be marked on slips of paper and transferred to the map, as, in diagram 32 (*b*), A, B, C, D and E were transferred to the central meridian, F, G, H, I and J to meridian 50° and so on.

Curves are drawn through the plotted points.



NORMAL ZENITHAL EQUIDISTANT

Diagram 33

(*d*) **Normal Zenithal Equidistant.** The Normal Zenithal Equidistant is one of the simplest projections to construct. It follows from the definition of equidistance[1] that the Equator and the central meridian are divided into equal sections respectively by the meridians and parallels. The meridians pass through the Poles and the parallels divide the circumference or the map into equal parts.

Through the three points thus fixed for each meridian and parallel, arcs of circles are drawn.

The scale of zenithal equidistant projections, whether polar, normal or oblique, is 1·57 times its 'true' length, or 1·57 times the scale of the 'projected' sphere; for the radius of the circular map equals a quarter of the circumference of the sphere; i.e. $\frac{2\pi R}{4}$, which equals 1·57 R.

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[1] See page 17.

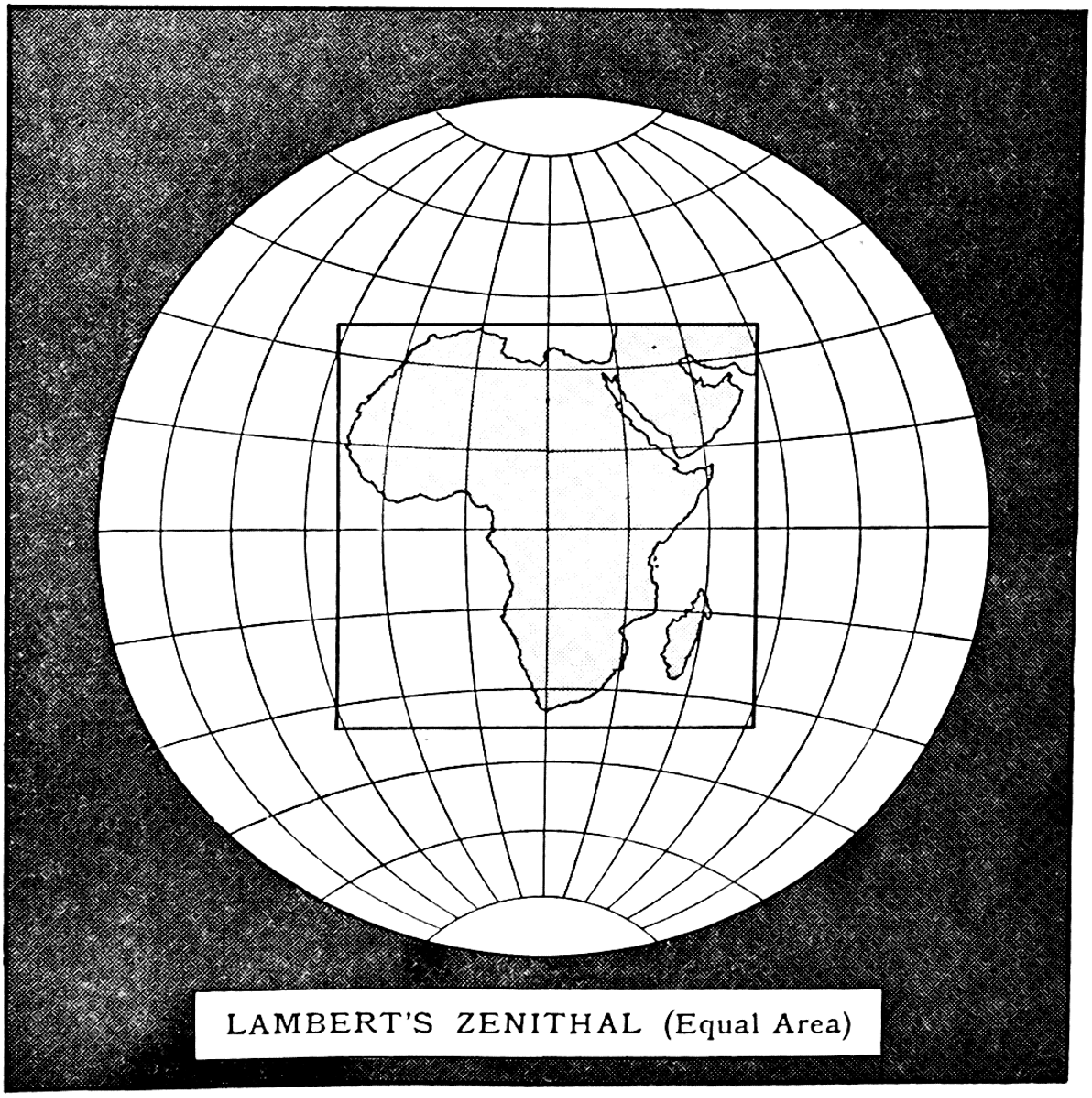
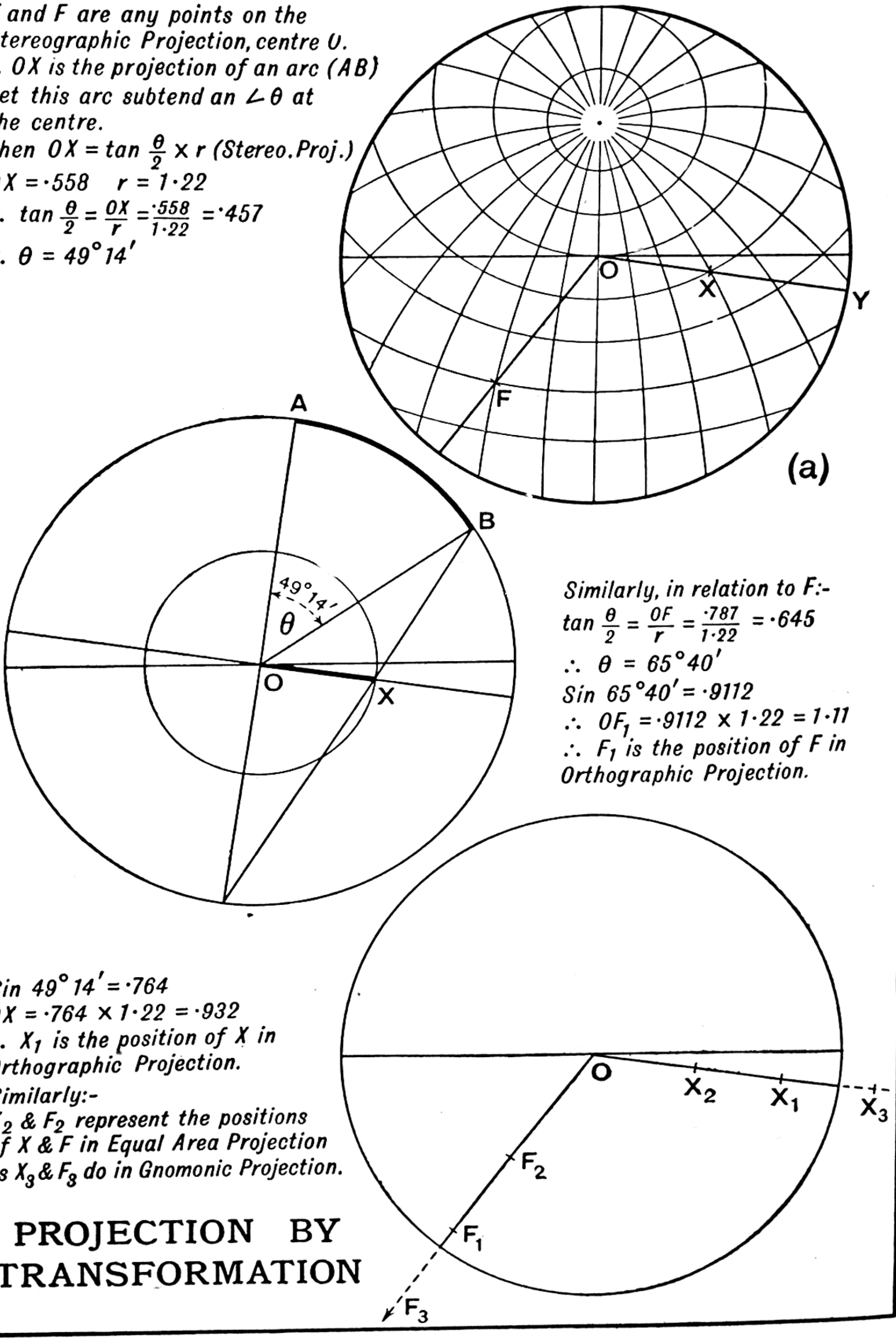


DIAGRAM 34

(*e*) **Normal Zenithal Equal Area.** The method of constructing a normal zenithal equal area map is similar to that of the equidistant, except that the Equator and central meridian are divided according to the scale of sin $z/2$ to a radius of $\sqrt{2}R$.

X and F are any points on the Stereographic Projection, centre O.
$\therefore OX$ is the projection of an arc (AB)
Let this arc subtend an $\angle\theta$ at the centre.
Then $OX = \tan\frac{\theta}{2} \times r$ (Stereo. Proj.)
$OX = \cdot558 \quad r = 1\cdot22$
$\therefore \tan\frac{\theta}{2} = \frac{OX}{r} = \frac{\cdot558}{1\cdot22} = \cdot457$
$\therefore \theta = 49^\circ 14'$



Similarly, in relation to F:-
$\tan\frac{\theta}{2} = \frac{OF}{r} = \frac{\cdot787}{1\cdot22} = \cdot645$
$\therefore \theta = 65^\circ 40'$
$\sin 65^\circ 40' = \cdot9112$
$\therefore OF_1 = \cdot9112 \times 1\cdot22 = 1\cdot11$
$\therefore F_1$ is the position of F in Orthographic Projection.

$\sin 49^\circ 14' = \cdot764$
$OX = \cdot764 \times 1\cdot22 = \cdot932$
$\therefore X_1$ is the position of X in Orthographic Projection.
Similarly:-
X_2 & F_2 represent the positions of X & F in Equal Area Projection as X_3 & F_3 do in Gnomonic Projection.

PROJECTION BY TRANSFORMATION

DIAGRAM 35

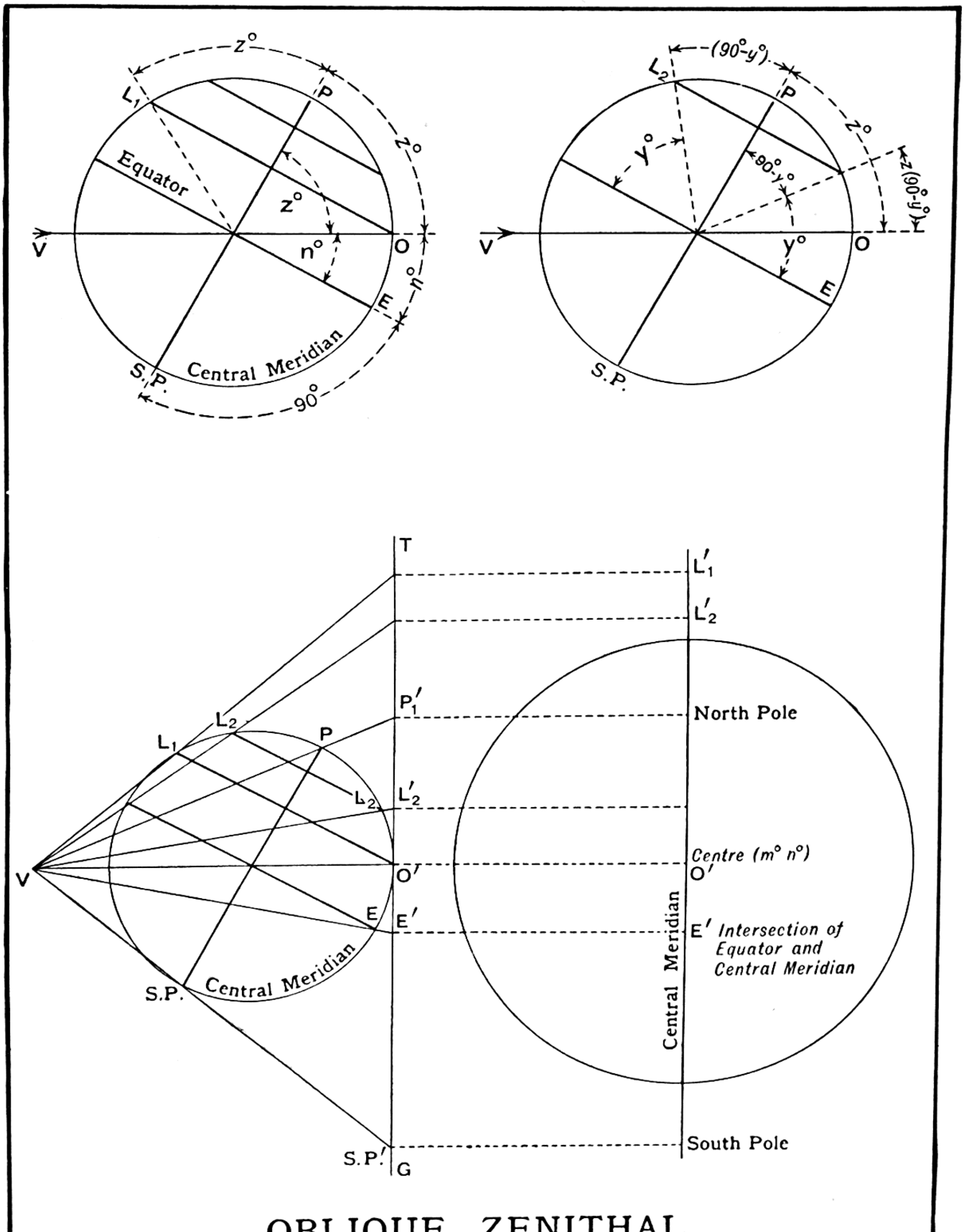
VI

OBLIQUE ZENITHAL PROJECTION, IN WHICH THE PLANE IS TANGENT TO THE SPHERE AT ANY POINT BETWEEN THE EQUATOR AND A POLE

General principle and projection by transformation. In polar zenithal projections we observed that the meridians radiate from the centre as straight lines at their true angles, and that the projections differ only in the scale of radii used to construct the concentric circles which represent the parallels. On the assumption that the earth is a sphere it is obviously possible to assume through any point the radiation of straight lines representing great circles through that point and smaller circles concentric at the point, representing any desired distances along the directional great circles radiating from that point.

It follows that any zenithal projection can be transformed into any other zenithal projection by the drawing of straight lines from the centre O through all intersections of meridians and parallels and by marking off new distances along these lines according to the appropriate scale.

In diagram 35 (*a*) let O be the centre of the projection and X any point, say, the intersection of meridian m° and parallel n°. OX produced meets the circumference at Υ. In any other projection with the same azimuthal property X will lie on a line making the same angle with the horizontal diameter as does $O\Upsilon$. If the given projection is Stereographic, then OX equals $R \tan \frac{\theta}{2}$. The angle θ can then be found from a table of natural tangents. $R \sin \theta$ would give the correct distance for OX in an orthographic, $R \tan \theta$ in a gnomonic and $R \sin \frac{\theta}{2}$ in an equal area zenithal projection. As many intersections of meridians and parallels as are desired can be found in this way and the meridians and parallels can be drawn through them.



OBLIQUE ZENITHAL

Diagram 36

Other general characteristics

(1) As the sphere is symmetrical about its axis, the central meridian will appear as a straight line vertically bisecting the map. It is convenient to regard this central meridian as 0°, as the meridians are arbitrarily numbered, and to regard the central point, therefore, as the intersection of parallel $n°$ and meridian 0°.

(2) If the selected parallel is $n°N$, the North Pole will appear on the central meridian, between O and the circumference, and the other meridians will be curves passing through the projection of the Pole. (The apparent exception is, of course, the Gnomonic, where the Poles are infinitely remote.)

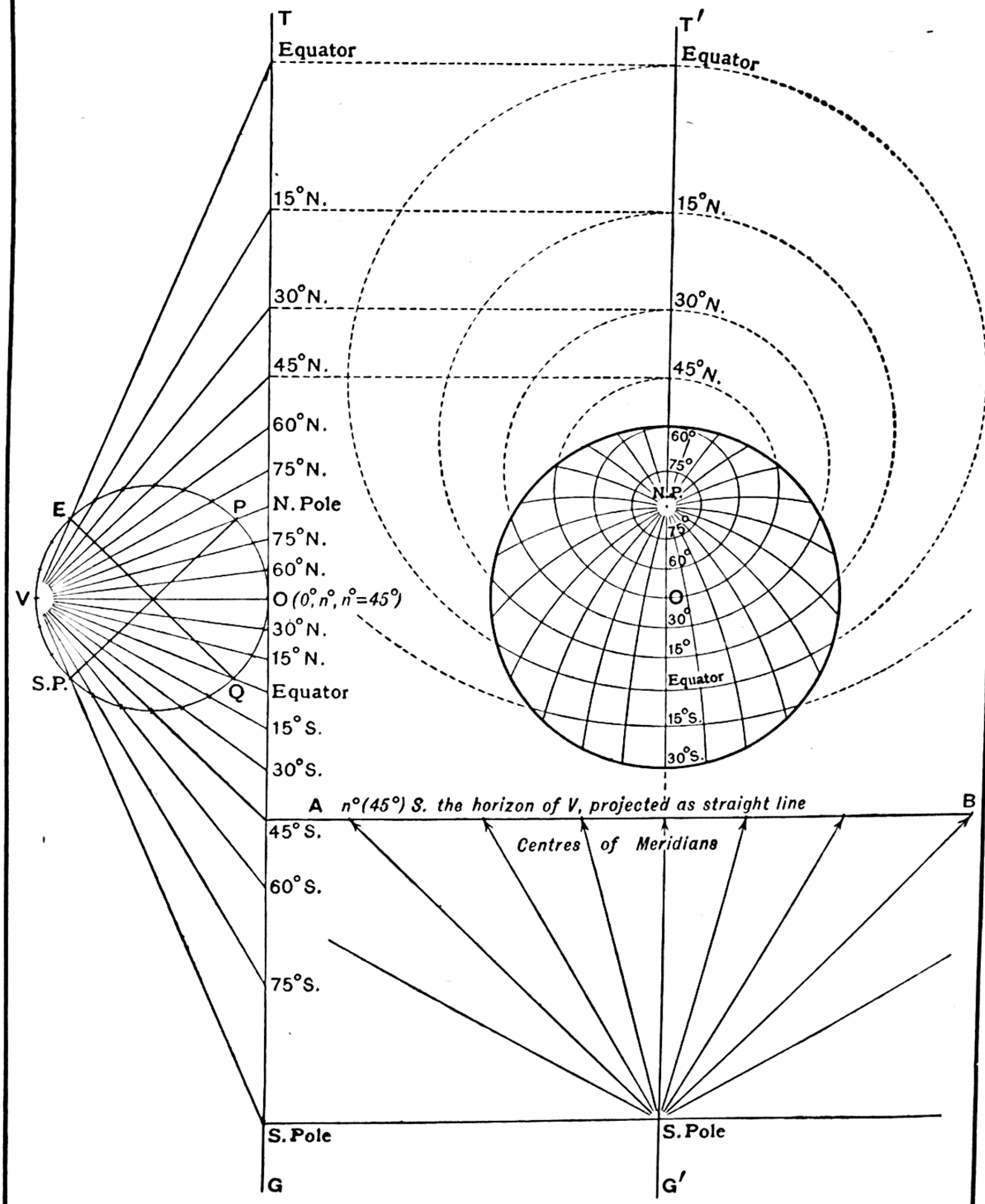
(3) It is clear from diagram 36 that :

$O'P'$ represents the projection of a circular arc which subtends an angle of $z°$ (or $90-n)°$ at the centre of the sphere ;

$O'E'$ represents the projection of a circular arc which subtends an angle of $n°$ at the centre of the sphere ;

Similarly, the angle subtended by the arc from O to the South Pole is $(90+n)°$;

the angle subtended by the arc from O to the intersection of any parallel $y°$ and the central meridian is $z°+(90-y)°$ if beyond P, and $z°-(90-y°)$ if between P and O.



STEREOGRAPHIC OBLIQUE ZENITHAL
(Graphic Method)

DIAGRAM 37

(*a*) **Stereographic Oblique Projection by Direct Method.** In diagram 37 *O*, the central point, is taken as the intersection of parallel 45° and the zero meridian. *TG*, the tangent at *O*, represents the projection of the central meridian, the circumference of the circular section of the sphere.

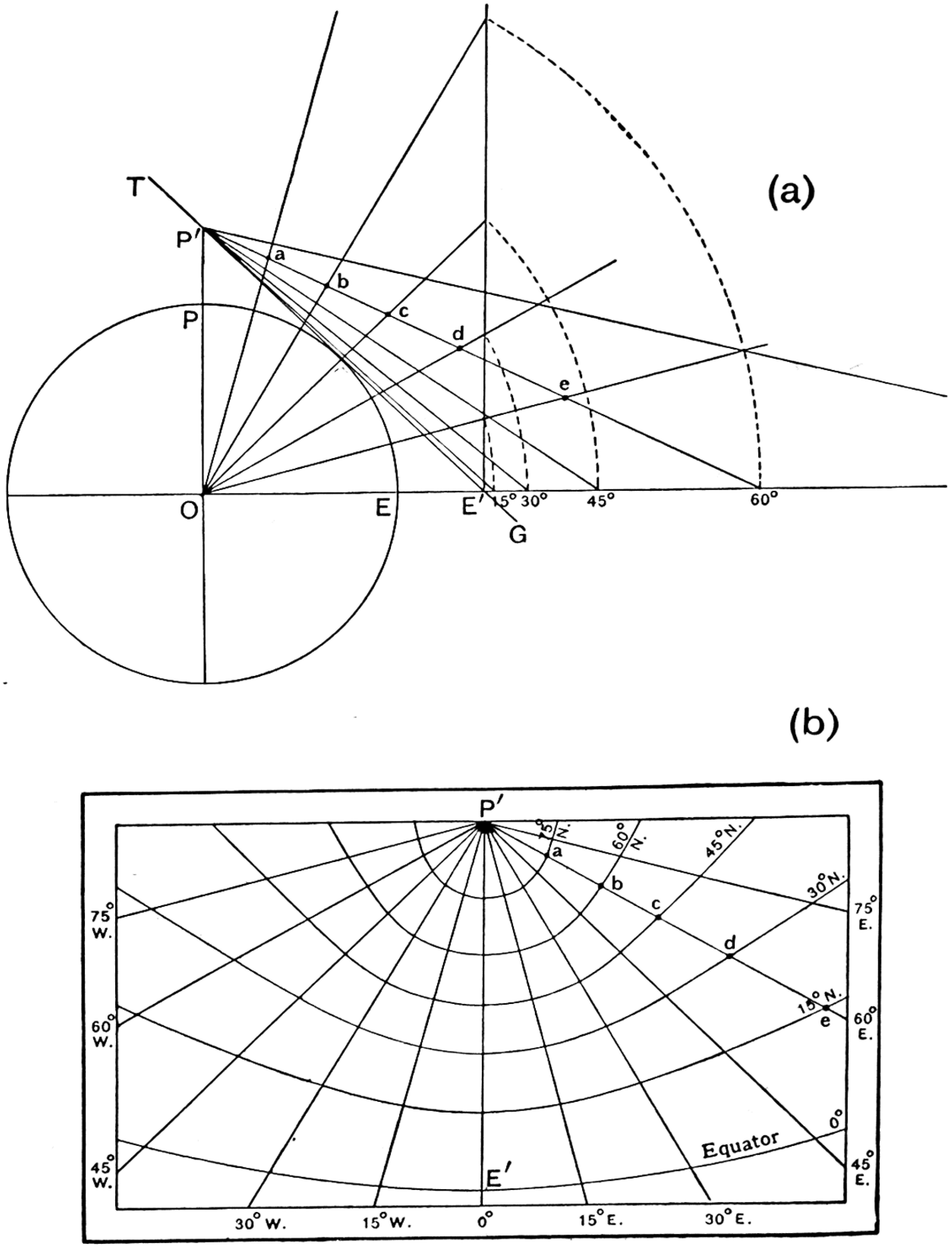
The Parallels. Lines radiating from *V* through the intersections of the parallels with the circumference and produced to meet *TG* give by definition the stereographic projections of these intersections on *TG* and these points can be transferred to *T′G′*, the central meridian of the map.

For each parallel there will be two such intersections and the centre of the circle representing any parallel will obviously lie midway between these two intersections of *TG*.

The Meridians. As all meridians pass through the Poles and are projected as arcs of circles, their centres lie on the perpendicular bisector of the line which joins the poles as projected along *TG*. In diagram 37, therefore, they lie along *AB*.

The centres also lie, as in normal projection, on lines perpendicular to the projected tangents at the Poles. The intersections of these normals with *AB* give, therefore, the required centres.

The circular boundary of the completed projection in diagram 37 is arbitrarily chosen, as much more than a hemisphere can be shown by this projection. There is no reason why the boundary should not have been a rectangle.



GNOMONIC OBLIQUE ZENITHAL

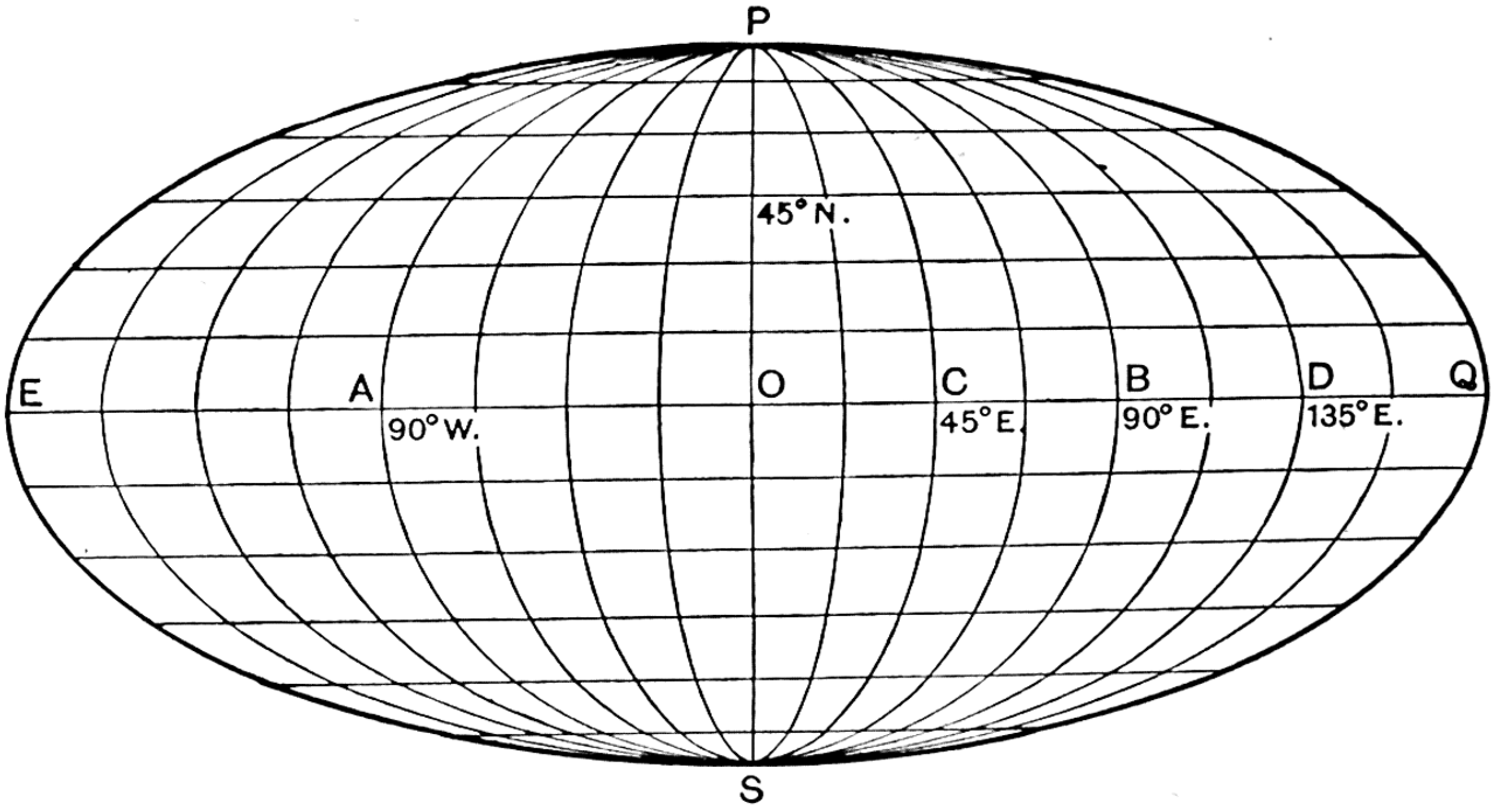
Diagram 38

(*b*) **Gnomonic Oblique Zenithal by Direct Method.** The differences between Gnomonic Oblique and Gnomonic Normal Zenithal are those due to the fact that the Pole appears on the projection in oblique zenithal instead of being infinitely remote as in normal gnomonic. Its position in relation to the central point will be found as in stereographic projection, except that the appropriate scale, that of tangents, will be employed.

The meridians, being great circles, and therefore projected as straight lines in all gnomonic projections, will radiate from the projected Pole at their true angles.

The intersections of the parallels with the meridians will be found as in normal gnomonic projection, except that the tangent plane (*TG* diagram 38 (*a*)) instead of being parallel to the central axis, is inclined to it at the required angle. It intersects *OP* produced at *P′*, the projection of the Pole, and *OE* at *E′*, the projection of *E*, the intersection of the Equator with the central meridian. These two points are first marked on the central meridian as in diagram 38 (*b*).

Secants of the angles of the parallels (every fifteenth in the diagram) are marked along *OE′* produced, to the radius *OE′* (instead of *OE* as before). The points so found are joined to *P′*. Now the procedure is similar. Radii from *O* are produced, numbered in accordance with the angle they make with *OE′*, and their intersections with the lines from *P′* to the points marking the secants are transferred to the map to give the required intersections of the meridians and parallels. The transference of the points *a*, *b*, *c*, *d* and *e* is shown along the meridian of 60° E., in diagram 38 (*b*).



MOLLWEIDE'S EQUAL AREA
Elliptical Projection

DIAGRAM 39 (*a*)

VII

IMPORTANT MODIFIED AND ARBITRARY PROJECTIONS

(*a*) **Mollweide's Equal Area Elliptical Projection** resembles Cylindrical Equal Area in that the parallels are projected as horizontal straight lines. Their exaggerated lengths towards the Poles, however, are avoided, as the meridians are shown as ellipses passing through the Poles.

In diagram 39a, *EQ* represents the Equator ; *PS*, the central meridian, equals half *EQ*. *O* is their point of intersection. *PBSA* is a circle and represents the meridian of 90° E. and W. All other meridians, including, therefore, the meridian which constitutes the boundary of the map, are represented by ellipses which divide the Equator into 360 equal parts.

PCS and *PDS* are halves of two such ellipses, bisecting *OB* in *C* and *BQ* in *D* respectively.

$$\frac{\text{Area of an ellipse of one axis } x}{\text{Area of a circle of diameter } x} = \frac{\text{other axis } y}{\text{diameter } x};$$

$$\therefore \frac{\text{Semi-ellipse } PCS}{1} = \frac{\text{semicircle } PBS}{2} = \frac{\text{semi-ellipse } PDS}{3}$$

$$= \frac{\text{semi-ellipse } PQS}{4};$$

∴ The figures included between the arcs are equal in area ;

∴ The meridians thus constructed cut off equal areas.

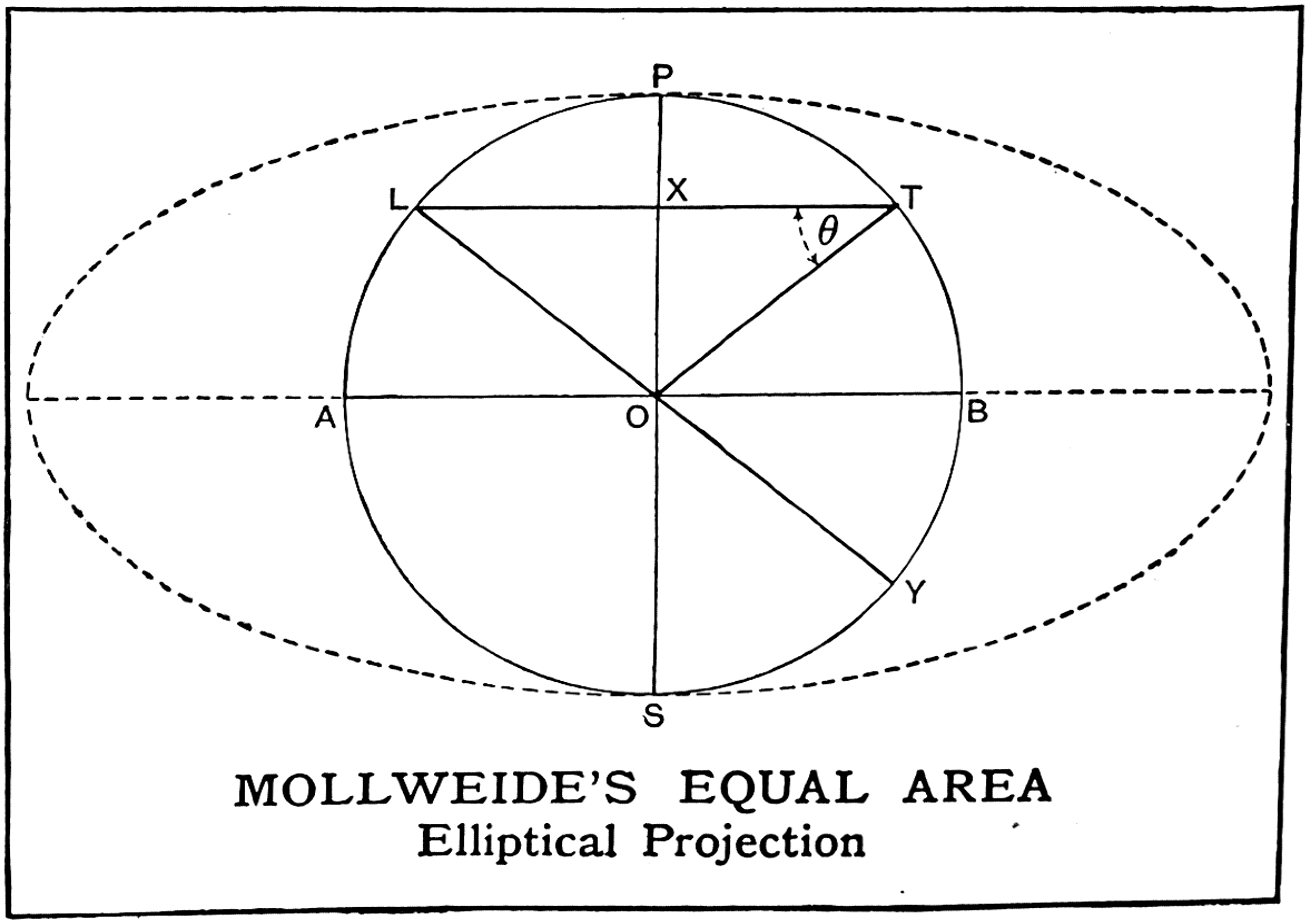


Diagram 39 (*b*).—The Spacing of the Parallels in Mollweide's Projection

To space the parallels so that they cut off areas equal to the corresponding belts on the sphere.

In diagram 39 (*b*) circle *PBSA* corresponds to the projection of meridian 90° in diagram 39 (*a*). *LT* is the projection of parallel $n°$.

Area of belt $LTBA$ = area of $\triangle LTO$ + area of sector TOY

$$= R^2 \sin\theta\cos\theta + R^2\theta$$

$$= \tfrac{1}{2}R^2\sin 2\theta + R^2\theta = \tfrac{1}{2}R^2(\sin 2\theta + 2\theta)$$

Area of corresponding surface on globe $\frac{1}{2}\pi R^2 \sin n°$.

$$\therefore\ \sin 2\theta + 2\theta = \pi\sin n°.$$

Sin θ gives the required distance.

Given θ it is easy to find $n°$, and from a sufficient number of cases to construct a table of distances. Such a table is included in the Appendix, p. 108.

In Diagram 39 (*b*) the area of the sector TOY would be expressed as $\dfrac{\pi R^2\theta}{180}$, instead of $R^2\theta$, if θ were considered in terms of degrees instead of radians. The formula relating θ and n would then be:

$$\sin 2\theta + 2\theta\frac{\pi}{180} = \pi\sin n°.$$

The relation between this formula and the table of distances on page 108 is made immediately apparent by such an example as the following:

Given that a parallel of latitude is 25·72 *mm. from the Equator in a map of Mollweide's Projection, drawn to a scale of* 1 : 100 *million* ($r = 63{\cdot}66$ *mm.*), *find by calculation the latitude of the parallel.*

Distance when radius is regarded as unity

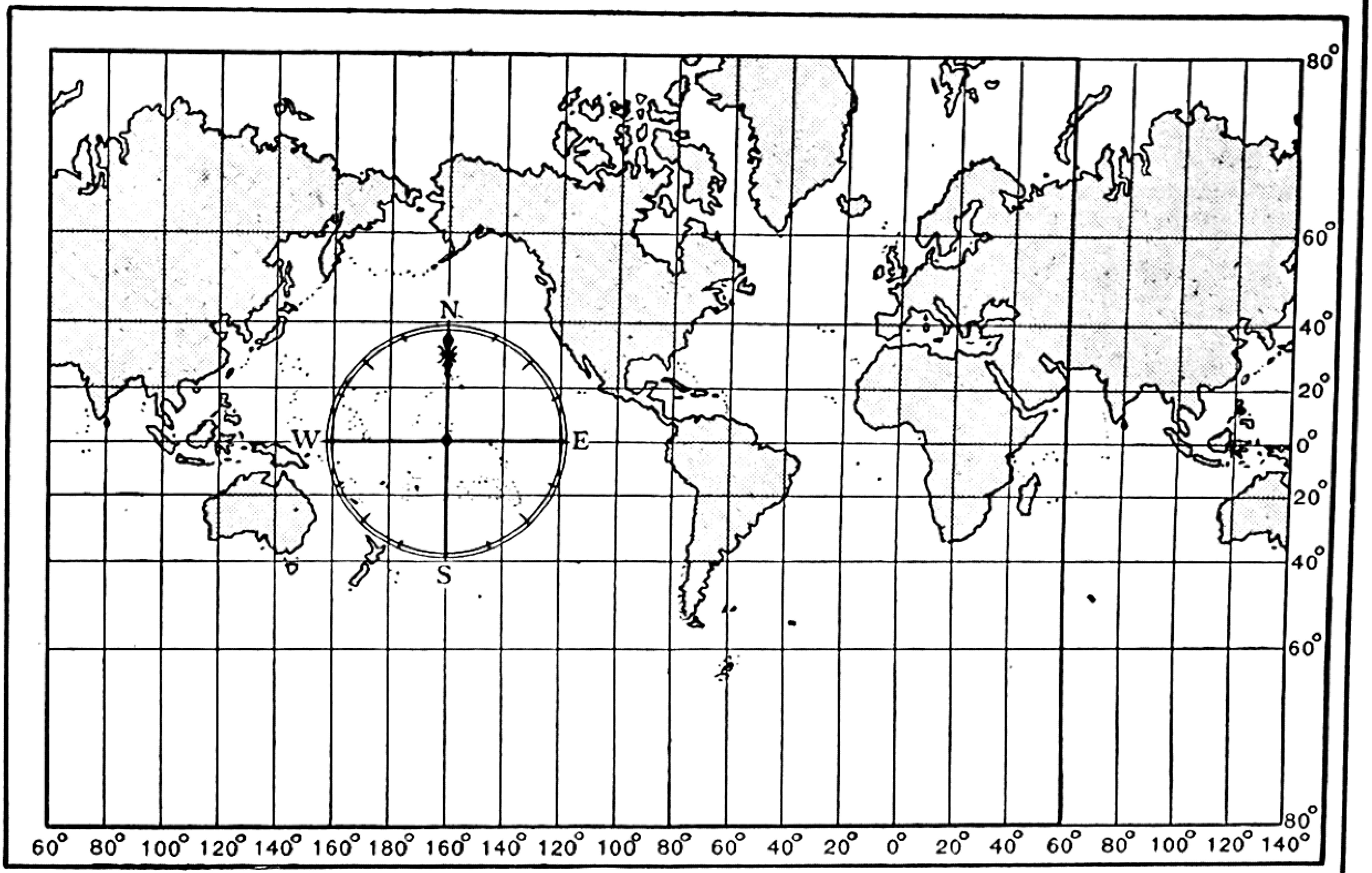
$$= \frac{25{\cdot}72}{63{\cdot}66} = 0{\cdot}404 = \sin\theta.$$

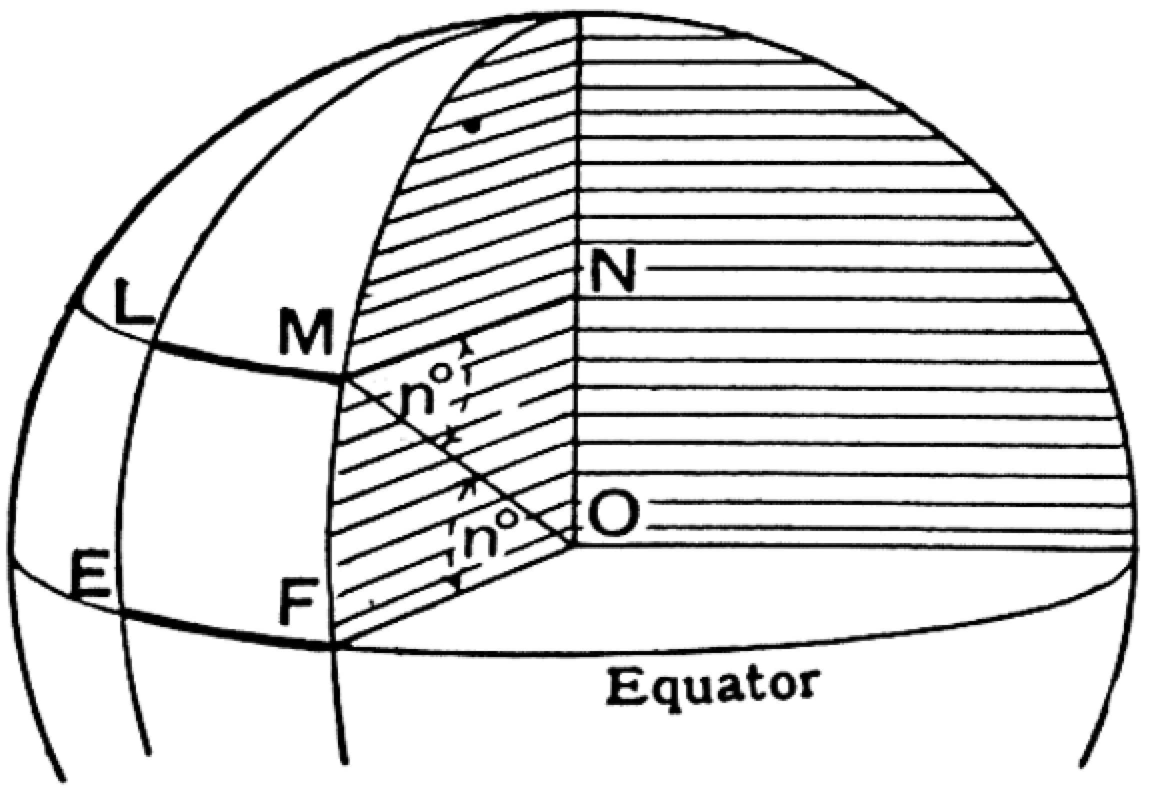
$$\therefore\ \theta = 23°\,50';$$

$$\pi\sin n° = \sin 2\theta + 2\theta\frac{\pi}{180} = 0{\cdot}74 + 0{\cdot}83 = 1{\cdot}57;$$

$$\therefore\ \sin n° = \frac{1{\cdot}57}{\pi} = 0{\cdot}5.$$

$\therefore$ the parallel is of latitude 30°.





(a)

Let m° of longitude be represented by EF at the Equator, and LM at n° of latitude.

Then, $\frac{LM}{EF} = \frac{MN}{FO} = \frac{MN}{R} = \cos n^\circ$

$\therefore\ LM = EF \cos n^\circ$

i.e. a degree of longitude at n° latitude = a degree of longitude at the Equator × $\cos n^\circ$.

MERCATOR'S PROJECTION

Diagram 40

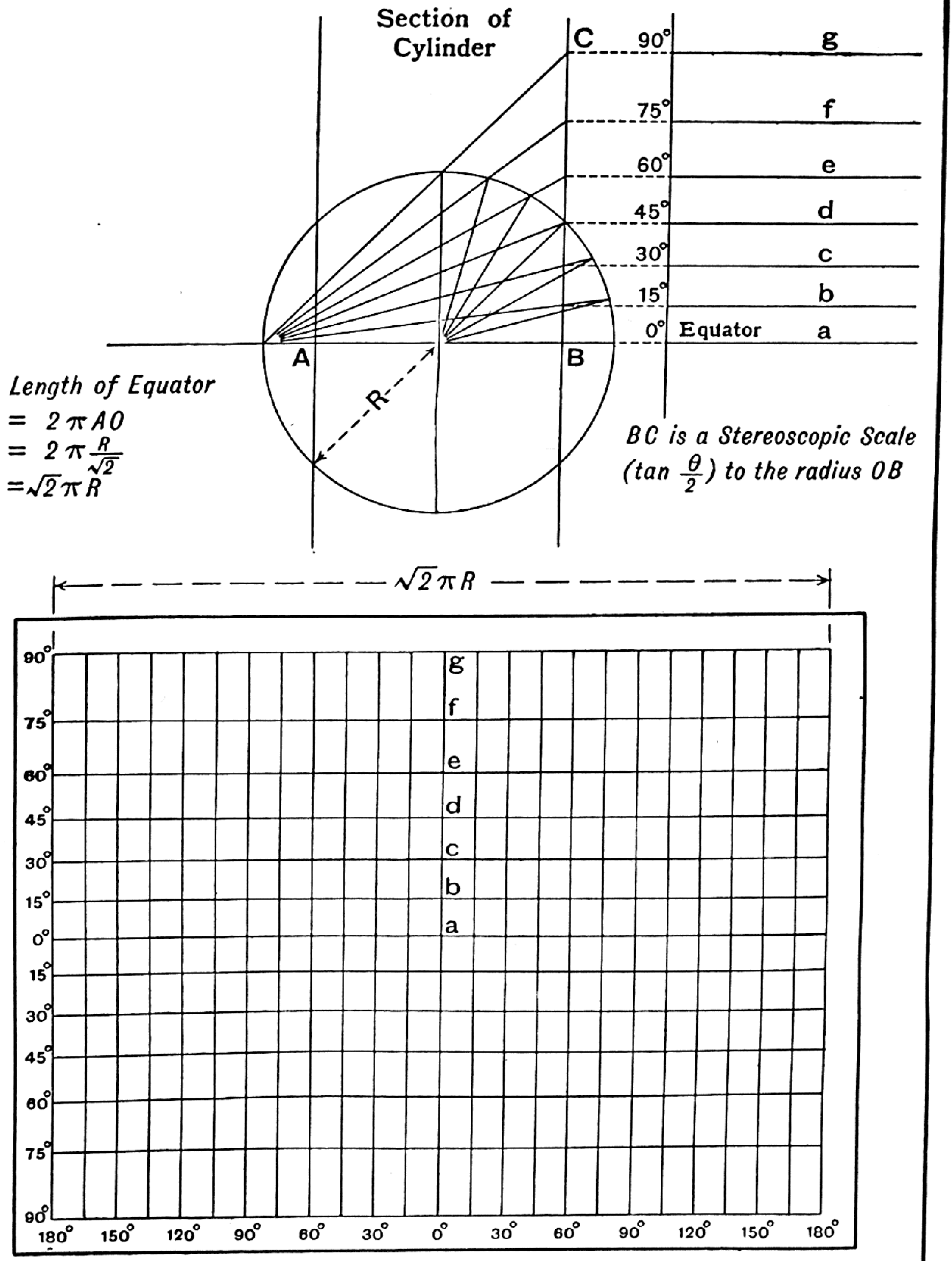
(*b*) **Mercator's Projection (Cylindrical Orthomorphic).** As Mercator's is a cylindrical projection, the meridians are shown as parallel lines perpendicular to the Equator and dividing it into 360 equal parts. As the distance between the meridians is, therefore, constant, whereas on the globe it decreases and finally vanishes as the Poles are approached, it follows that the same scale can be maintained along meridians and parallels (i.e. the map can be orthomorphic) only if the distances between the parallels are increased in the same proportion as the true distances between the meridians have been increased by the fact of their projections being parallel.

It is clear from diagram 40 that the true distance between two meridians at any latitude $n°$ equals the length of a degree of longitude at the Equator multiplied by cos $n°$ (or divided by sec $n°$). On Mercator's Projection, therefore, this distance is sec $n°$ times its true length. *Therefore, the distance between the parallels must be so increased that a degree of latitude, at n degrees of latitude, is represented by a distance of sec n° times its true length.* This true length is represented by a degree of longitude at the Equator, which is equivalent to a distance of $\frac{\pi R}{180}$.

This increase is, of course, continuous, showing a proportionate increase for every minute as well as for every degree, so that the distance of any parallel from the Equator represents the sum of an infinite series. There is no very simple way of showing the derivation of the formula which expresses this distance, which, for the parallel of $n°$ of latitude is :

$$R \log_e \tan\left(45 + \frac{n}{2}\right)^{\circ}.$$

A table showing the distance from the Equator of every fifth degree, (*a*) with the radius as unity and (*b*) with the radius as 63·66 mm. is appended. (See page 108.)



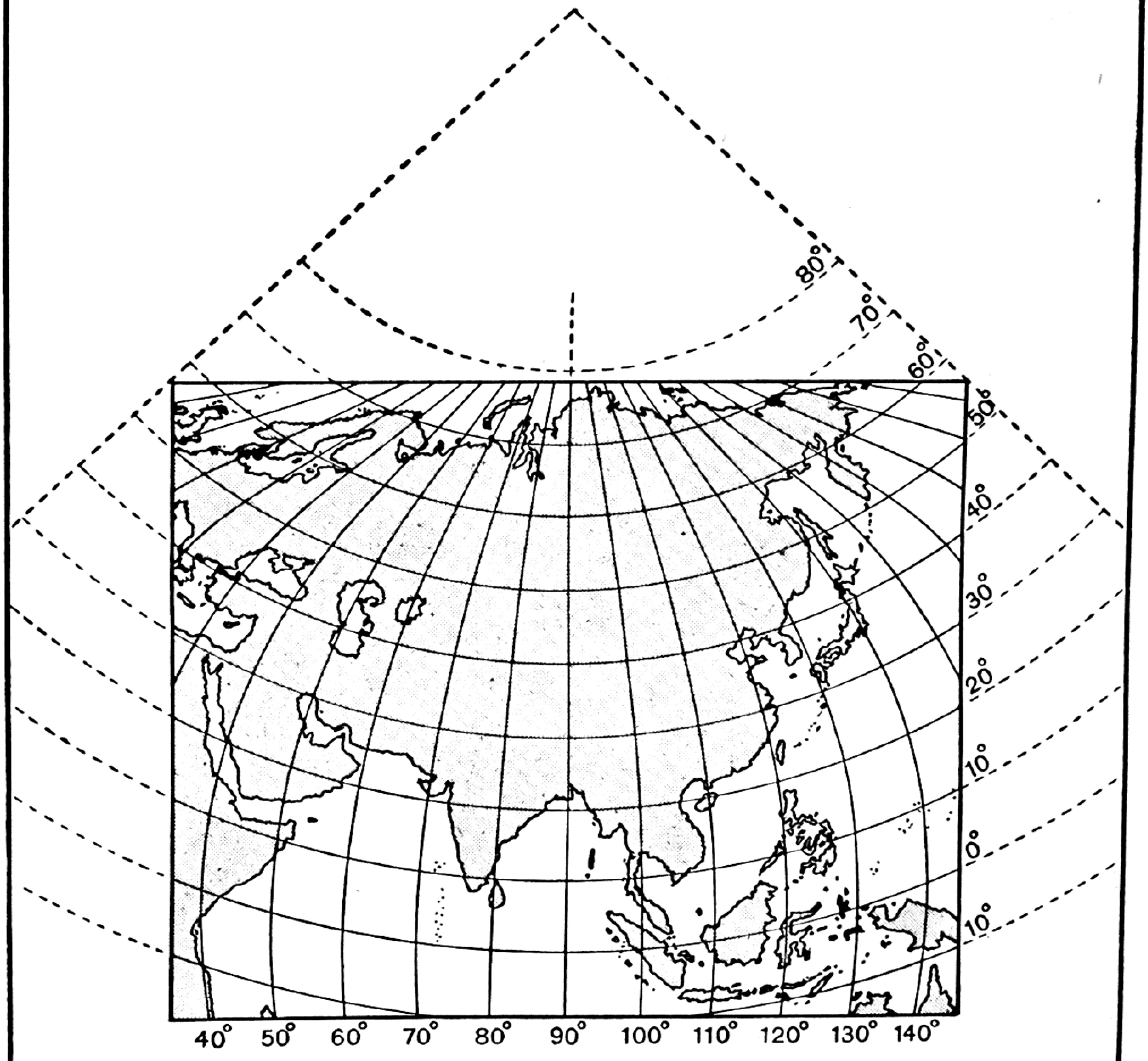
GALL'S PROJECTION

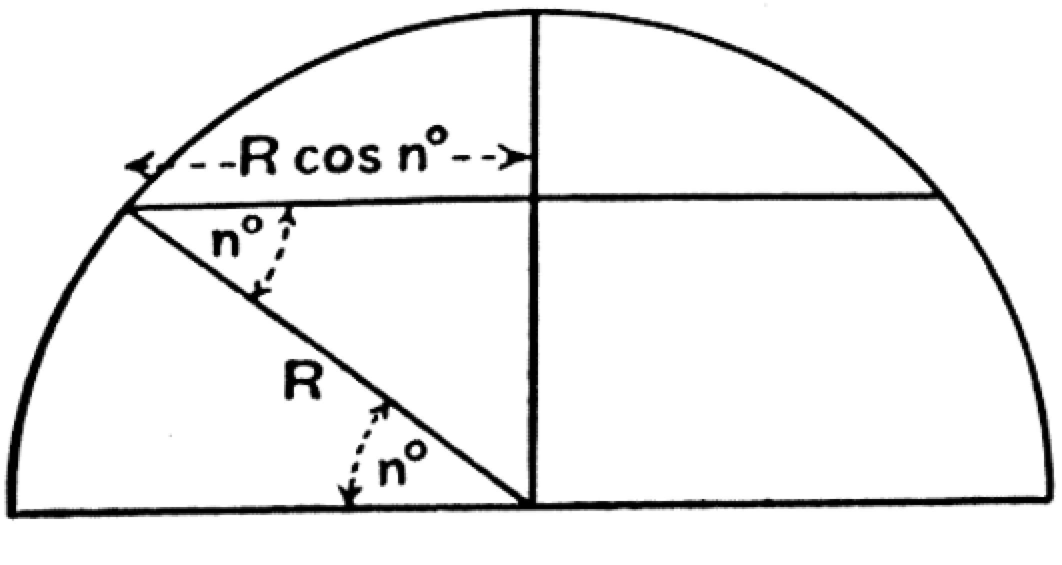
DIAGRAM 41

(*c*) **Gall's Stereographic Cylindrical Projection.** Gall's Projection is a cylindrical projection very similar to Mercator's, but it forfeits Mercator's orthomorphic quality by the reduction of the distance between the parallels in order to avoid the fantastic exaggeration of areas and distances which cylindrical orthomorphism produces towards the polar regions. Consequently, northern and southern regions look crushed.

The map is conceived as being projected on to the surface of a cylinder which cuts through the sphere at parallels 45° N. and 45° S. ; i.e. halfway between the Equator and the Poles. The meridians, as in all cylindrical projections, are shown as equally spaced vertical straight lines. The parallels are projected stereographically, i.e. from a point on the sphere opposite to the surface being mapped.

The projection is one of the simplest to construct and can be graphically carried out except for the calculation of the length of the Equator. From diagram 40 it is clear that AO equals $R \cos 45°$ and, therefore, that the diameter of the cylinder equals $\sqrt{2}R$. The length of the Equator, therefore, equals $\sqrt{2}\pi R$.





Circumferences of Parallels, and, ∴, lengths of equally intersected arcs, vary as their radii; i.e., as cos latitude.

BONNE'S PROJECTION

DIAGRAM 42

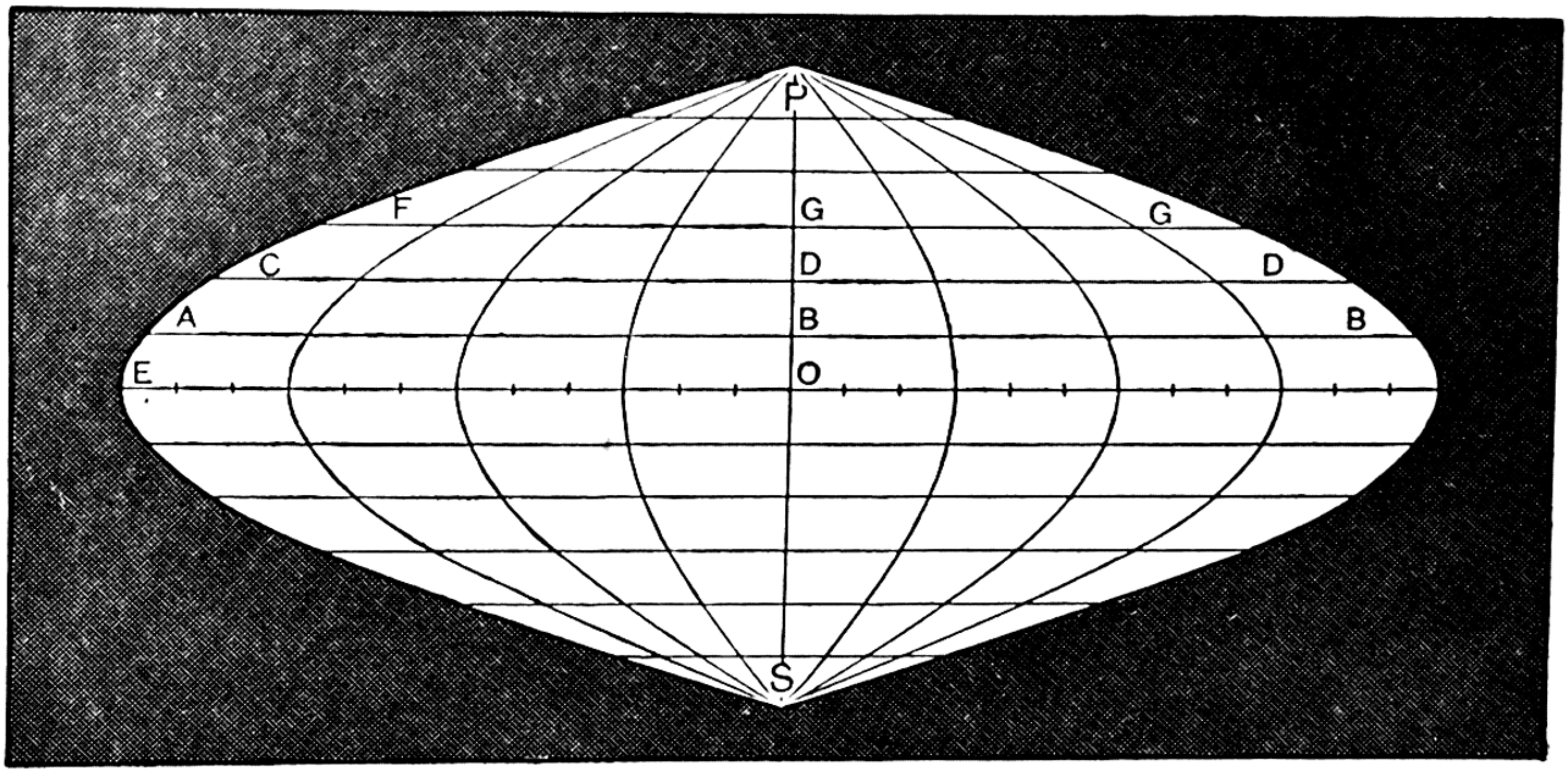
(*d*) **Bonne's Modified Conical Projection.** Bonne's useful projection is not really a conical projection at all, as the meridians are not shown as radiating straight lines but as curves. The central meridian only is shown as a straight line, divided equally to show correct distances.

Through these points concentric arcs are constructed. These are then divided truly, so that the correct scale of distance is shown along every parallel instead of along one or two only as in conical with one or two standard parallels.

This projection is an equal area projection. A small area on the globe, the small rectangle between two consecutive parallels and two consecutive meridians, will be represented on the map by a small parallelogram on the same base and between the same parallels and will, therefore, be of equal area.

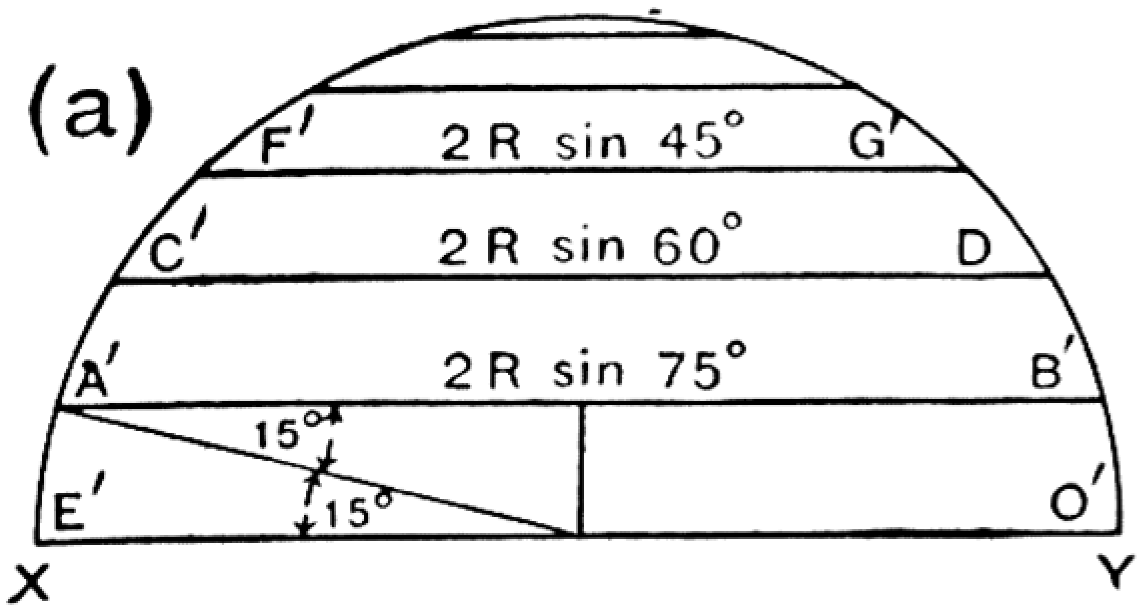
A simple graphical method of construction, in which on a small scale map the error is negligible, consists in marking along the Equator, with a pair of dividers, distances equal to the divisions of the central meridian, and, if this division be called x, in dividing every other parallel with intervals equal to $x \cos n^\circ$. The distances will then be the lengths of the chords intersected by the meridians instead of the lengths of the arcs intersected; but if every tenth degree is shown the error will be less than 1/10,000.

Through the points thus plotted the meridians are drawn.



(b)

(a)



$E'O' = EO, \quad A'B' = AB$ etc.
$EO = PS$

SANSON-FLAMSTEED or SINUSOIDAL EQUAL AREA PROJECTION

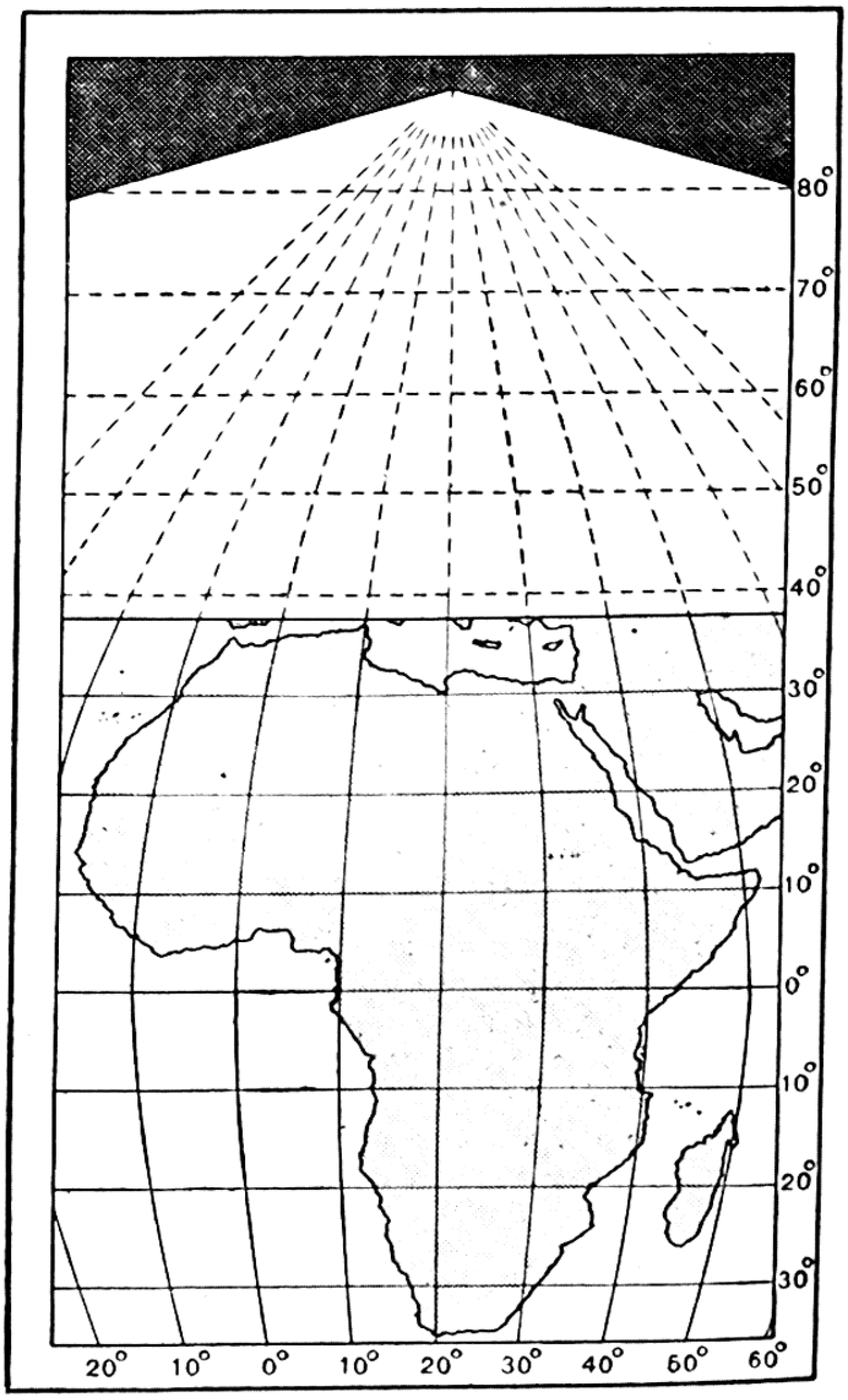


DIAGRAM 43

(*e*) **The Sanson-Flamsteed or Sinusoidal Equal Area Projection.** The Sanson-Flamsteed Projection is a specific case of Bonne's, with the Equator as the standard parallel. The Equator and all parallels are represented by parallel straight lines, intersecting the central meridian at right angles and at their true distances. The meridians are marked along the parallels at their points of intersection at their true distances and are, therefore, sine curves, as is clear from diagram 43.

Diagram 43 (*a*) represents an orthographic projection of every fifteenth parallel, in a circle of which the diameter XY equals EO, half the Equator in the projection, diagram 43 (*b*). On either side of the central meridian distances are marked along the parallels equal to the orthographically projected parallels. Through the points obtained the boundary curves are drawn. The other meridians cut off equal divisions along the parallels. The parallels will obviously be of lengths equal to $4\,R \sin z^\circ$ and, as the circumferences of circles are proportional to their radii, will be of their true relative lengths.

SECTION THREE

THE PROJECTIONS APPLIED

THOUGH map projection is interesting purely as a study, and valuable in that even an elementary knowledge of it tends to modify with advantage one's attitude towards an atlas, one of the most interesting questions in map projection yet remains : What particular projections should we choose if confronted with the task of constructing an atlas? It will be useful first to recapitulate the essential characteristics of each class of projections, so that any particular projection may be readily recallable and immediately recognisable.

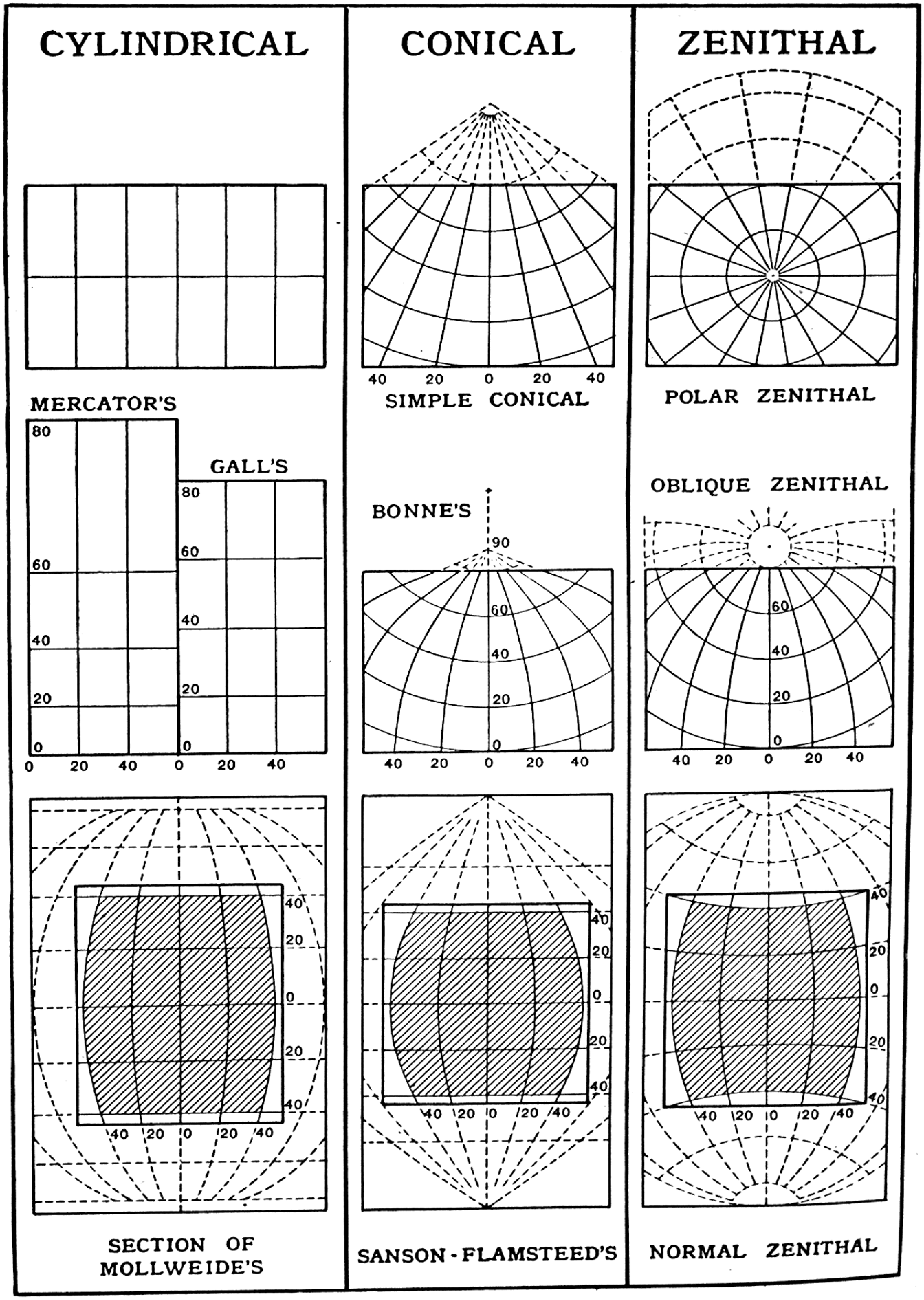


Diagram 44.—Characteristics of the Projections

I

CHARACTERISTICS OF THE PROJECTIONS

OMITTING the Polar Zenithal Projections, which are restricted in application and unmistakable in appearance, we can recognise the rest in general use by the following characteristics :

(1) If the meridians are parallel straight lines the projection is *cylindrical* ;

(2) If the meridians are converging straight lines it is *conical*, almost certainly with two standard parallels ;

(3) If the meridians are curved, the projection is

Bonne's if the parallels are concentric circular arcs,

Zenithal if the parallels are other curves,

Sanson-Flamsteed's if the parallels are equidistant straight lines, and

Mollweide's if the parallels are straight lines approaching nearer to each other towards the Poles.

The most useful Zenithal Projections are Lambert's, which combines equal area with a reasonable preservation of shape over extensive regions, and the Zenithal Equidistant which, while closely resembling Lambert's, keeps still more accurately to shape without any undue distortion of area. Stereographic Projection, like all orthomorphic projections, retains accuracy of shape for small areas at the cost of an increasing exaggeration of area towards the more remote regions. This implies that large regions are, in fact, considerably distorted in shape, and though the projection is occasionally used for hemispheres it has little general usefulness other than ease of construction. Orthographic Projection, with its infinite perspective, has little practical value except for the construction of star charts, and the Gnomonic is in general reserved for the special purpose of projecting great circle routes by straight lines.

While the choice between the zenithal projections and the selection of either Lambert's Equal Area or the Equidistant Projection are dependent on the qualities which are inherent in the projections themselves, the general applicability of the projections described in the previous pages can best be studied in relation to a number of typical problems in map construction. For this purpose maps may be most usefully divided into three groups : (1) maps of the world, (2) maps of large equatorial regions, and (3) maps of regions north or south of the Equator.

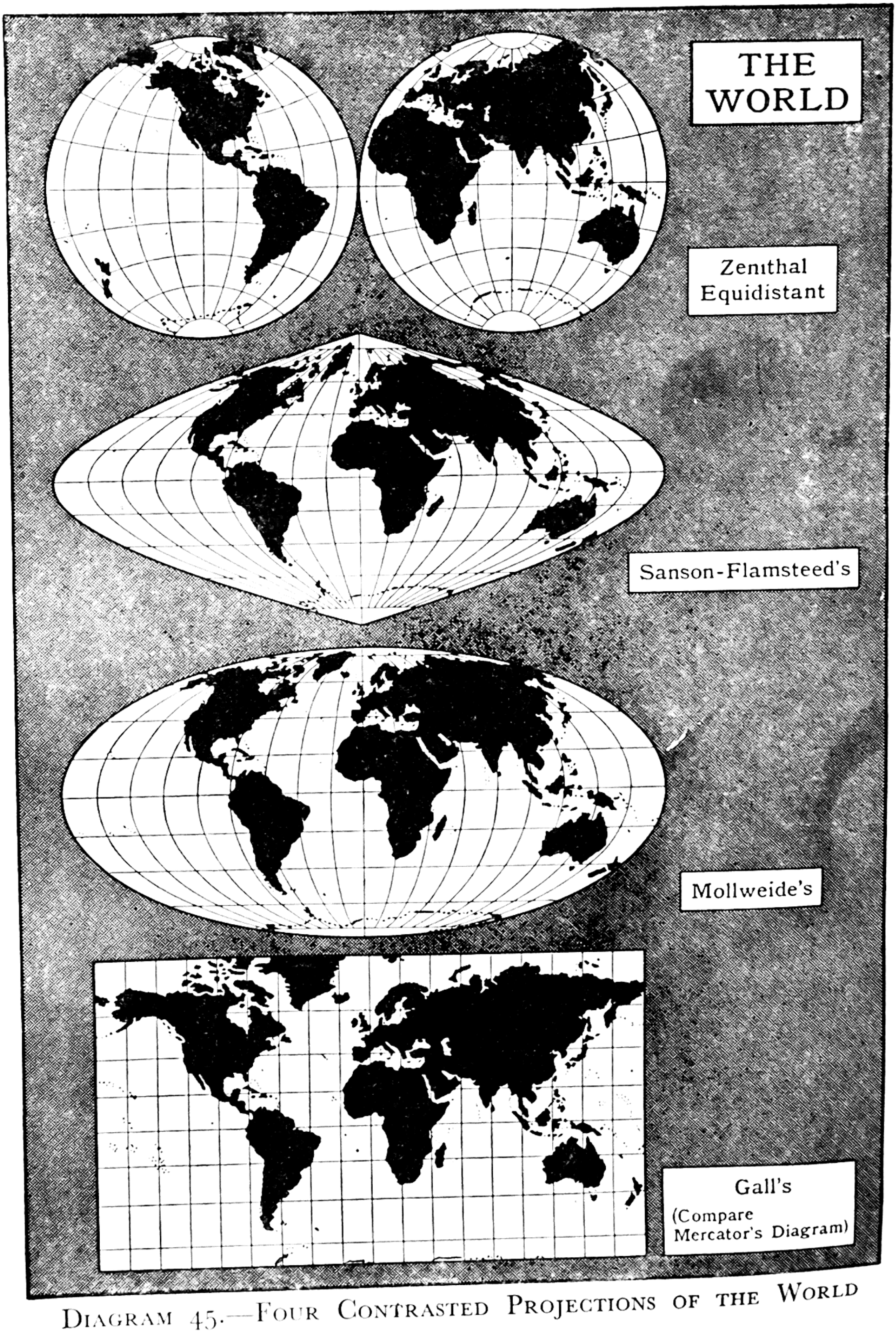


Diagram 45.—Four Contrasted Projections of the World

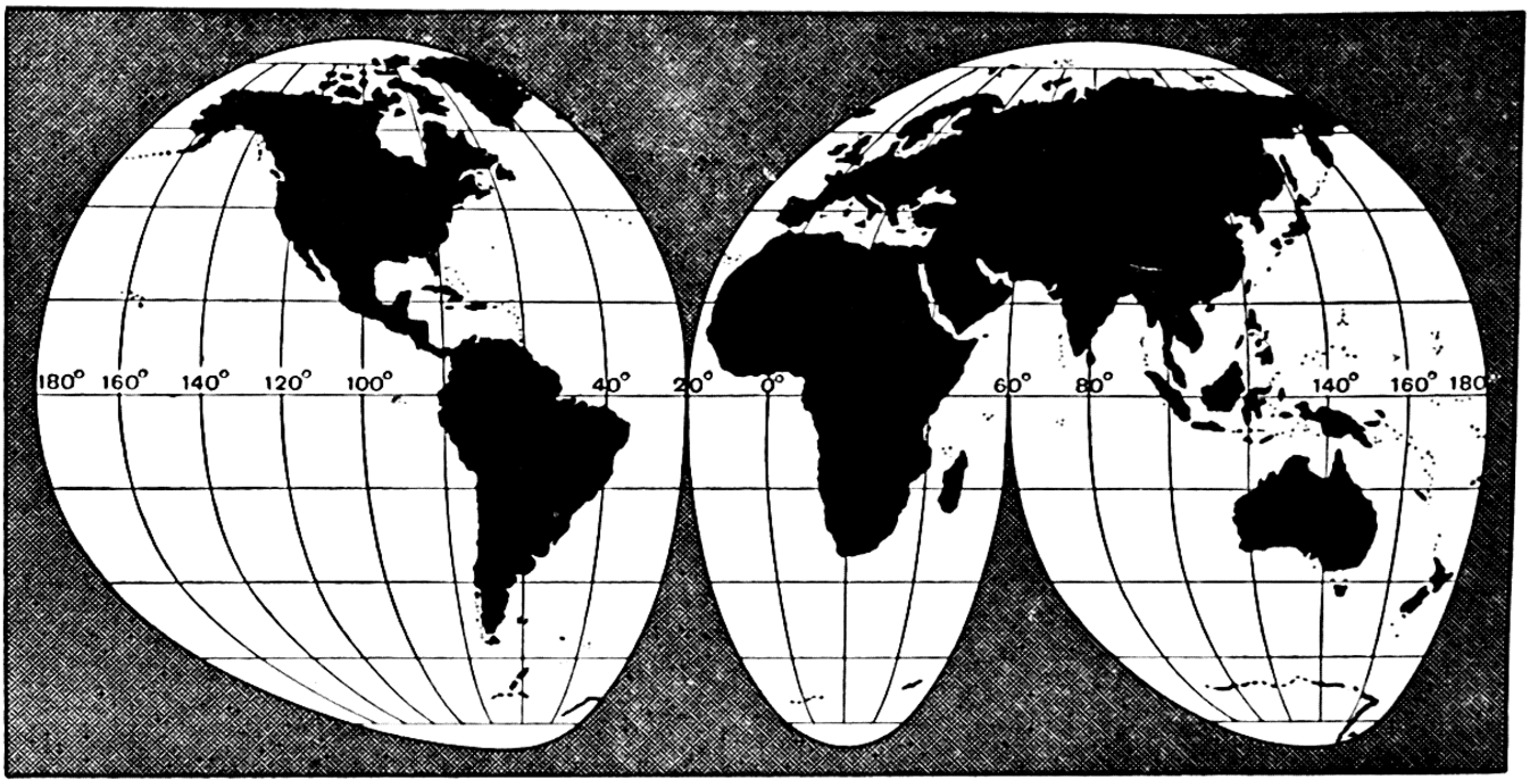
II

THE WORLD

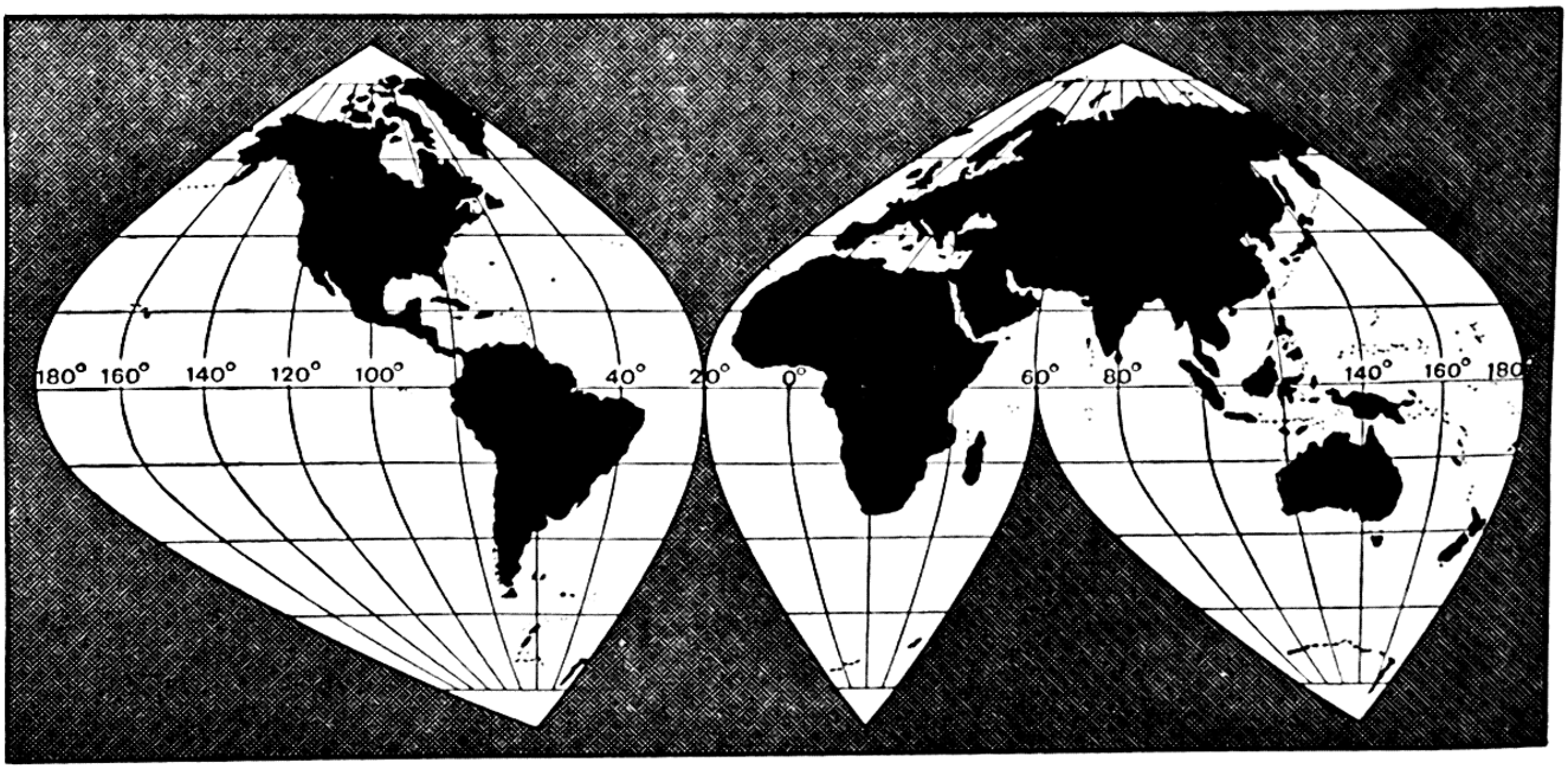
It is possible to represent the world either (1) as two hemispheres, in which case a Zenithal Projection is the obvious choice, (2) as a continuous map by using either a cylindrical projection, Mollweide's Elliptical or Sanson-Flamsteed's (45), or (3) by a modification of either of the last two projections, known as Re-centred or Interrupted Homolographic Projection (46). The disadvantages and advantages of these respective methods are apparent as soon as they are seen together.

To show the world in hemispheres is to construct not one but two maps, and in this fact lie both its advantages and disadvantages in comparison with other methods. Obviously there will be less distortion in a map which represents only half the globe than in a map which attempts to represent it all. On the other hand it is often desirable to show the world as a continuous surface, as for example in those maps which show ocean currents, wind systems, isotherms, storm tracks and the like.

If a continuous map of the world is needed, we can reject the ordinary Sanson-Flamsteed Projection, of which the excessive distortion of shape towards the extremities of the map restricts its usefulness to the representation of less extensive regions. If the map ought to represent areas in correct proportion, as in all statistical or distributional maps, Mollweide's is the obvious choice. If it is more important to represent shape than area, and if the map must be continuous, we are left only with a cylindrical projection and the necessity of choosing between Mercator's, Gall's, or some compromise based on a modification of these. Mercator's has the somewhat doubtful advantage of being orthomorphic and the distinction of being the only projection which accurately represents compass bearings from all points by straight lines. Gall's, however, avoids some of the consequent distortion and is probably, in general, less misleading.



Based on Mollweide's



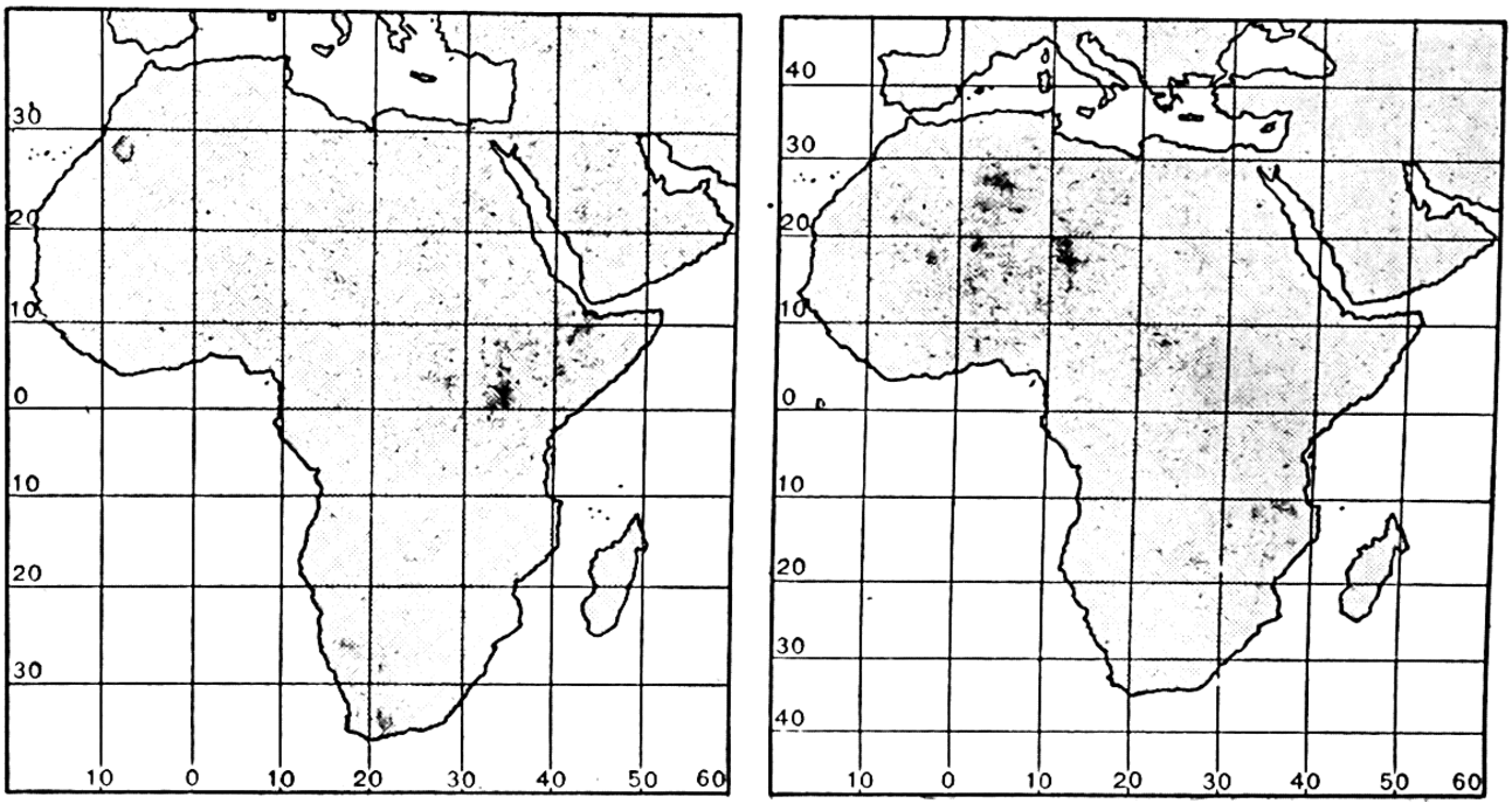
Based on Sanson-Flamsteed's

RE-CENTRED or INTERRUPTED HOMOLOGRAPHIC PROJECTION

DIAGRAM 46

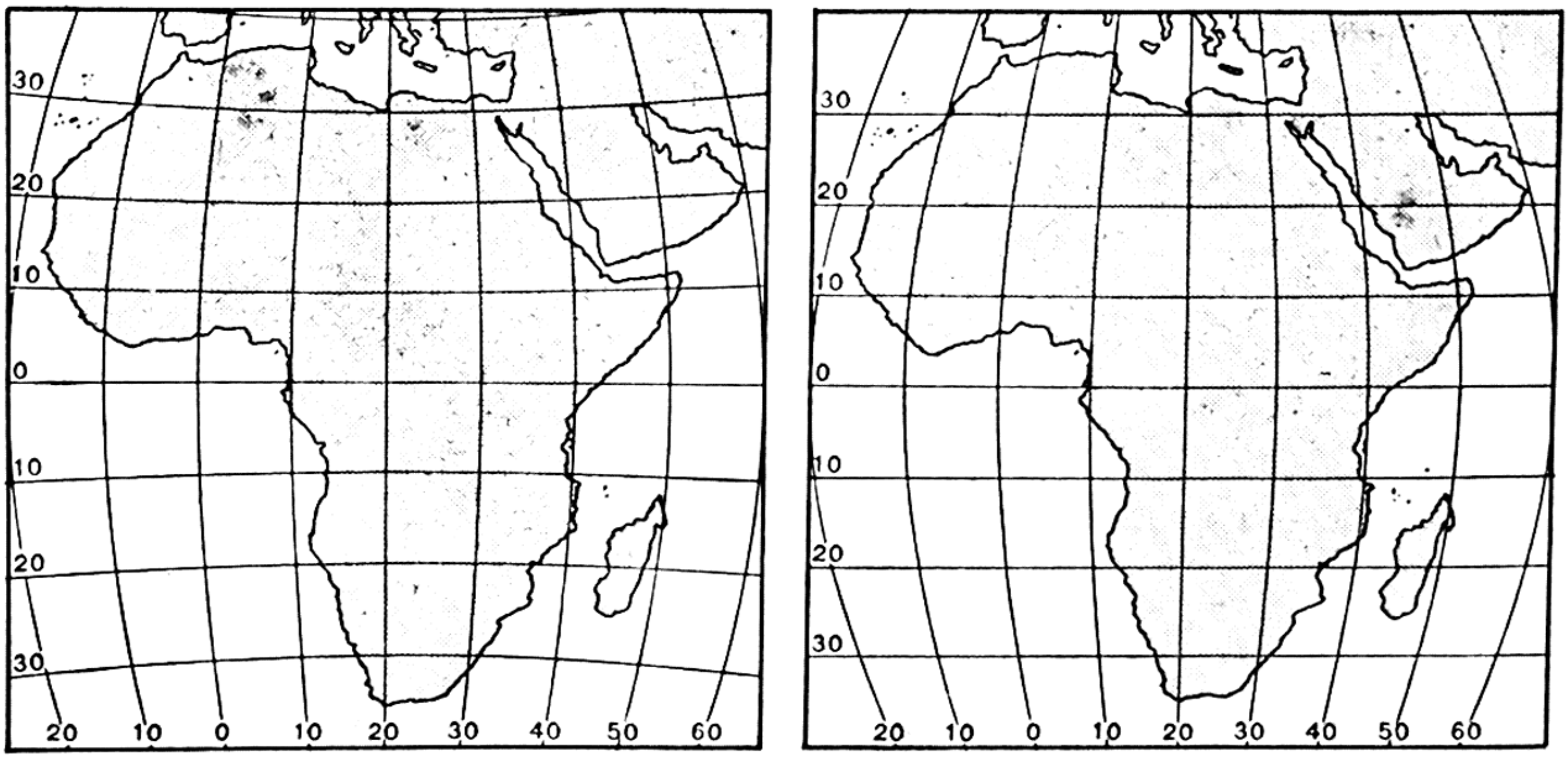
The third method, deservedly gaining popularity, may be regarded as a compromise between the continuous map of the world and that which represents it in hemispheres. As the latter employs two vertical central meridians about which each hemisphere respectively is constructed, the Interrupted Homolographic Projection employs a number of vertical meridians selected with reference to the great land masses of the world, so that these may be shown with the minimum distortion. The term 'homolographic', applied to map projection, means the representation of equal areas with minimum distortion; literally it means 'similarly drawn'. The Equator is divided and the parallels are spaced as in the ordinary projection either of Mollweide or of Sanson-Flamsteed. In the accompanying diagram the selected meridians are 100° W. and 80° E., north of the Equator, 60° W., 20° and 140° E., south of the Equator. These pass, respectively, through the centres of North America, the Eurasian land mass, South America, Africa and Australia. The intersections of the parallels and the other meridians are calculated as in continuous projections, but from these selected meridians instead of from one central meridian.

It is only the great lateral extension of the land mass of Europe and Asia which prevents this equal area projection from showing the whole land surface of the world without appreciable distortion. In 'The Oxford Advanced Atlas', Bartholomew, in a similar arrangement of Sanson-Flamsteed's Projection, selects 20° E., north of the Equator, which gives an almost perfect Europe and northern Africa, but involves great distortion of eastern and especially of north-eastern Asia. In an admirable school atlas produced by George Philip, a similar projection based on Mollweide's Elliptic employs 60° E., north of the Equator, and centres the whole of the New World round the meridian of 90° W. For most general purposes this kind of world map, whether based on Mollweide's or Sanson-Flamsteed's projection, is admirable.



CYLINDRICAL ORTHOMORPHIC
(Mercator's)

CYLINDRICAL EQUAL-AREA



LAMBERT'S ZENITHAL
Equal Area

SANSON-FLAMSTEED'S
Equal Area

FOUR PROJECTIONS OF AFRICA
TO SAME EQUATORIAL SCALE

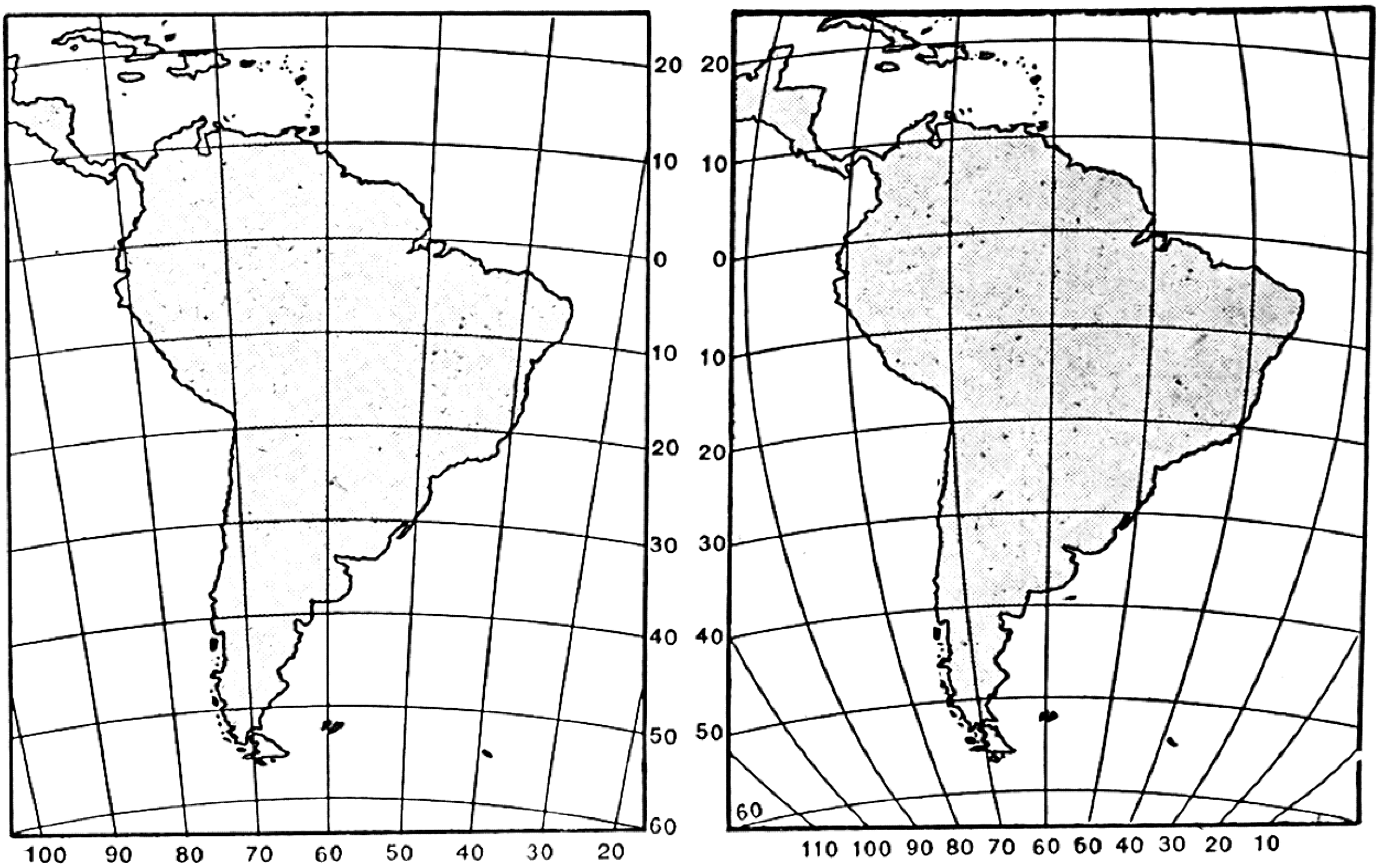
DIAGRAM 47

III

EXTENSIVE EQUATORIAL REGIONS

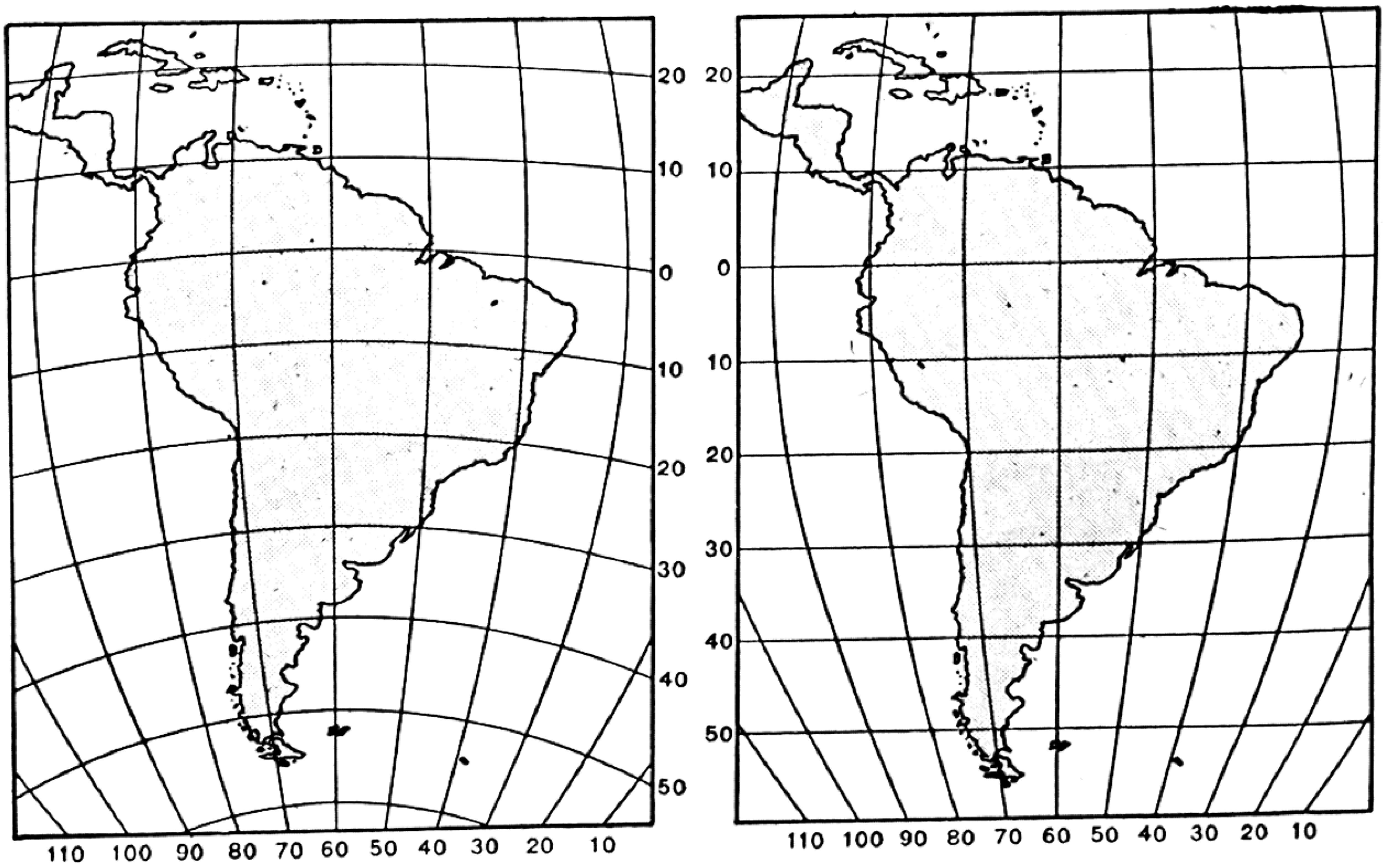
Before leaving Mollweide's, Sanson-Flamsteed's and the cylindrical projections, we have to consider their claims for the projection of those large equatorial regions of which Africa, spread almost equally across the Equator in length if not in area, is the best example. Other examples would include the Indian, Pacific and Atlantic Oceans, the Americas, Oceania, and South America. Another class of projections which have to be considered for such maps is the group of Normal Zenithal Projections, which show the Equator as a straight line across the centre of the map. Lambert's Equal Area and the Zenithal Equidistant have already been selected as the best representatives of this class.

The cylindrical projections, considered as map-nets for Africa, have all the disadvantages which they had when we considered them for the mapping of the world, without the necessity for our tolerating them ; for Sanson-Flamsteed's Projection combines good shape, for a region such as Africa, with accurately proportioned areas. Mollweide's, which is essentially a projection for world maps, achieves its superiority of shape round the world's edge by forfeiting it for specific regions ; it is a general compromise. The only serious rival to the Sanson-Flamsteed is Lambert's Equal Area Projection, and both are in general use.



CONICAL

BONNE'S



LAMBERT'S

SANSON-FLAMSTEED'S

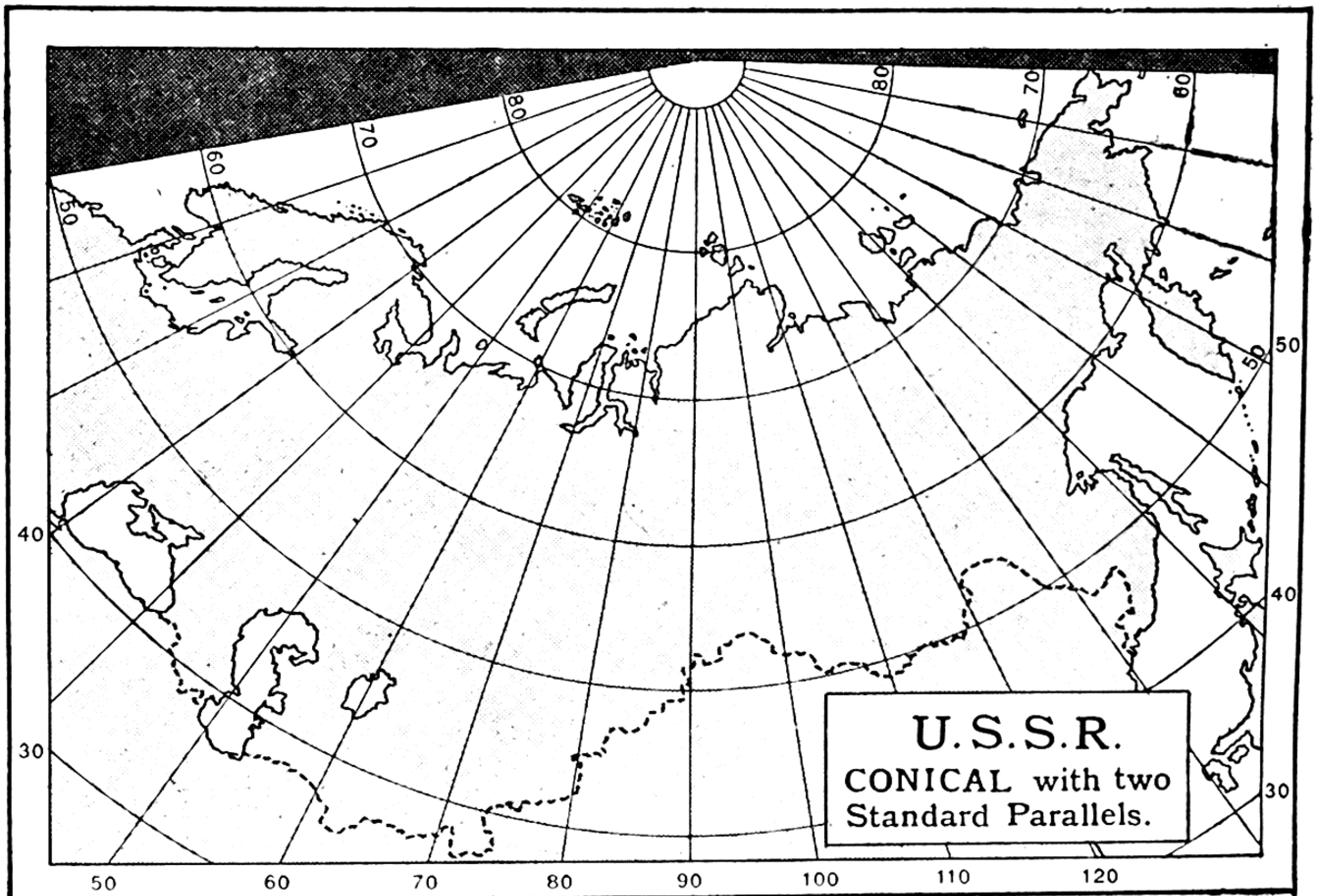
FOUR PROJECTIONS OF SOUTH AMERICA

DIAGRAM 48

South America differs from Africa, from the standpoint of map projection, in that the Equator, instead of dividing it centrally, divides it approximately in the proportion of one to five. As the Equator does cross it, however, both Sanson-Flamsteed's and Lambert's Projections are suitable, and, in fact, Sanson's produces an excellent map of South America.

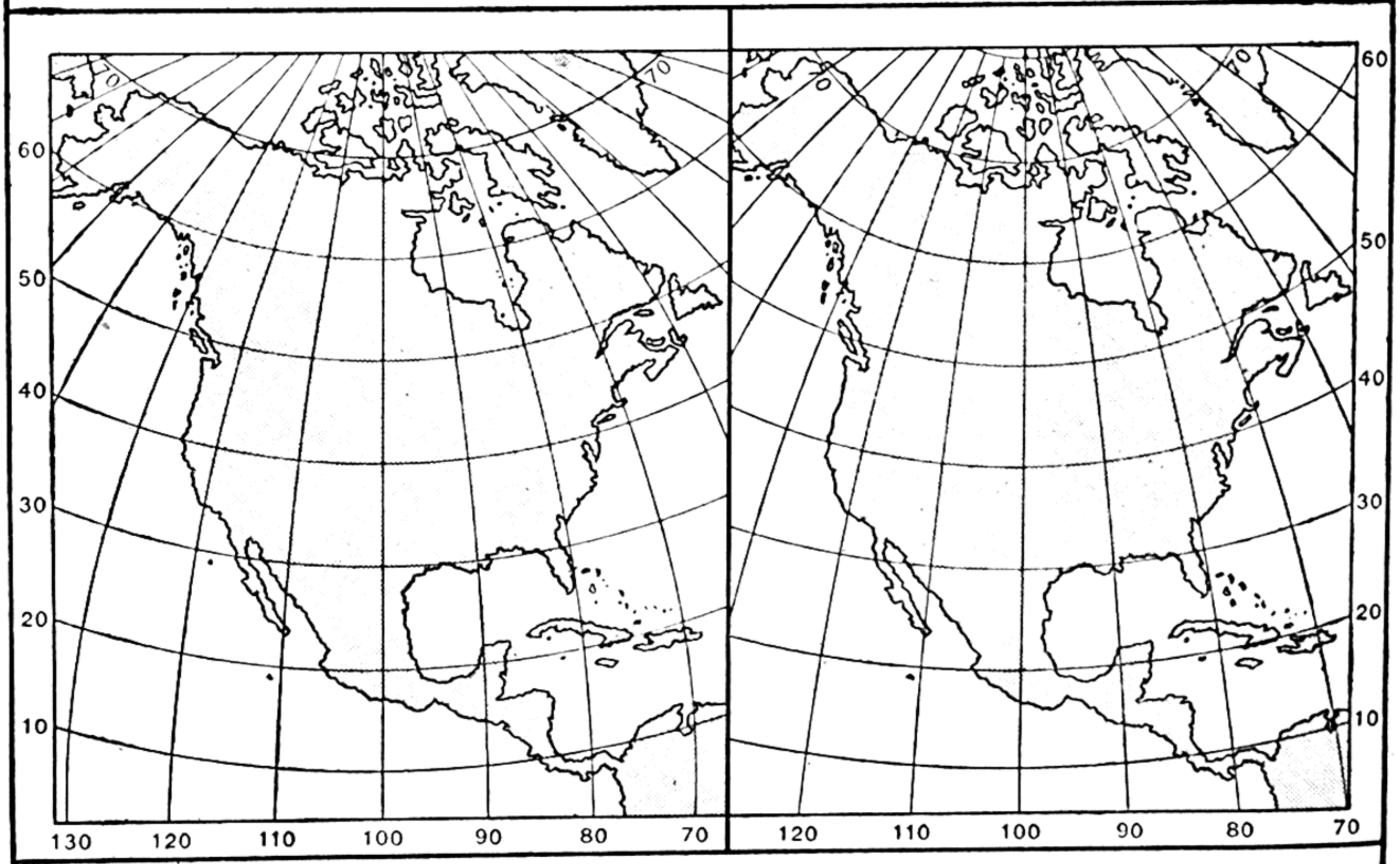
As so much of the country lies in the southern hemisphere, however, it introduces for the first time the possibility of using an oblique projection, either the Equidistant or Lambert's, or alternatively a conical or Bonne's. The great similarity between these projections is evident from the diagram. In selecting, firstly, between one of this group and, say, the Sanson-Flamsteed, the only point of importance is that in the latter the centre of interest, the area of greatest accuracy, lies along the Equator, whereas in the conical projection with two standard parallels, in Bonne's and in the oblique zenithals, the area of greatest accuracy could be arranged much more centrally in a map of South America than in the region of the Equator.

The essential difference between the three somewhat similar projections which remain, Bonne's, the conical and the oblique zenithal (whether Lambert's or the Equidistant) is that the extension of the area of greatest accuracy is distributed differently in each. This difference can best be examined in relation to those regions which are situated entirely, or almost entirely, to the north or south of the Equator.



U.S.S.R.

CONICAL with two Standard Parallels.



BONNE'S PROJECTION

LAMBERT'S PROJECTION

NORTH AMERICA

Scale along Central Meridian is similar in both maps.
Note effect of the widening parallels in Lambert's.

DIAGRAM 49.—PROJECTION OF REGIONS LYING NORTH OF THE EQUATOR

IV

REGIONS NORTH OR SOUTH OF THE EQUATOR

For the projection of a region wholly or almost entirely lying to the north or to the south of the Equator, it should be obvious now that the best choice lies between Bonne's, the conical with two standard parallels, or Lambert's Equal Area Projection. A glance through a number of good atlases will reveal that for many maps, as for example that of Europe, the choice is apparently immaterial. We have already indicated, however, that they are respectively characterised by the distributional extension of their greatest areas of accuracy.

All zenithal maps are projected from a point on the line which passes vertically through their centre. It follows that their regions of accuracy, or of decreasing accuracy, are concentric. They are ideal, therefore, for those regions which are ‘ as broad as long ’. The conical projection, with its rapidly increasing ‘ breadth ’ towards the base of the cone, is obviously better suited for those maps which take in considerable differences of longitude than those which extend principally from north to south. Bonne's projection, with its curved meridians, avoids this difficulty, but at the cost of increased distortion at the corners of the map. Whereas, for example, the three projections are almost equally good for a map of North America, the conical would be best for a map of Canada or for a map restricted to the U.S.A. The triangular formation of India suggests Bonne's rather than the conical, for the corners of the map are either outside the coastline or beyond the frontiers.

V

SUMMARY OF PROJECTIONS SPECIALLY SUITABLE FOR PARTICULAR MAPS

THE WORLD—	
(*a*) in Hemispheres	Polar Zenithal ; preferably Lambert's or the Equidistant
(*b*) Statistical	Mollweide's
(*c*) Currents, Isotherms, etc.	Gall's or Mollweide's
(*d*) Rhumb line Navigation	Mercator's only
(*e*) Great Circle Navigation	Gnomonic only
EQUATORIAL REGIONS—	
Africa, Central Africa, South America, Oceania, etc.	Sanson-Flamsteed or Lambert's
South America, Oceania	Bonne's also suitable
POLAR REGIONS	Lambert's or the Equidistant.
REGIONS NORTH OR SOUTH OF THE EQUATOR	Bonne's, Conical, Lambert's or the Equidistant Zenithal
If region is of wide lateral spread :	the conical is best ;
If region is relatively extended from north to south	Bonne's or Lambert's is preferable ;
If region is roughly diamond-shaped ;	Bonne's is ideal ;
If region is roughly circular or square in shape :	Lambert's or the Equidistant is preferable.

For small regions, as for European countries, there is little to choose between these four projections.

APPENDICES

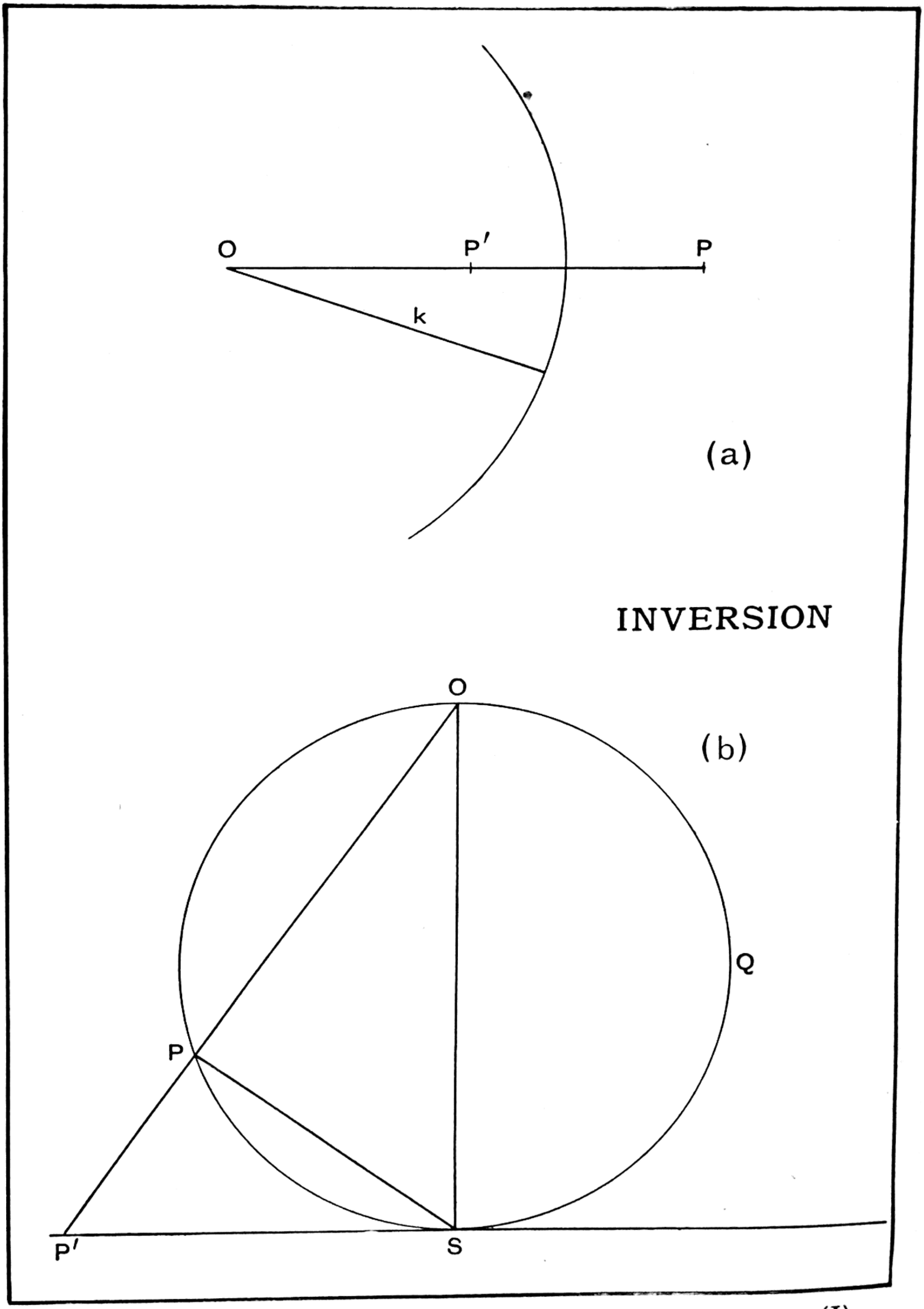


DIAGRAM 50.—STEREOGRAPHIC PROJECTION AS INVERSION (I)

APPENDIX A

STEREOGRAPHIC PROJECTION AS INVERSION

STEREOGRAPHIC Projection is a special case of the process known as *Inversion*, and the two important qualities of the projection (that circles which do not pass through the view point are projected as circles, and that angles are unaltered in magnitude by the projection) are fundamental properties of inversion.

Quite simply, inversion is this: if there are, in the same straight line, a fixed point O and two movable points P and P', which move so that $OP \,.\, OP'$ (the product of their respective distances from O) remains constant, then P and P' are inverse to each other in respect to O, 'the centre of inversion'.

Let P and P' approach each other until they meet at a distance k from O. Then $OP \,.\, OP'$ equals k^2; and k^2 is the constant product of $OP \,.\, OP'$ and is known as 'the constant of inversion'.

If P traces out a locus, to every position of P there will be a corresponding position of P'. The locus of P' is known as the inverse of the locus of P. Similarly the locus of P is known as the inverse of the locus of P'.

To show that Stereographic Projection satisfies these conditions

Let O, the fixed point, represent the view point, and P be any point on the sphere. The circle $OPSQ$ represents a section of the sphere containing the polar axis and $P'S$ the section of the tangent plane on which P' is the projection of P.

Join PS.

Triangles $OP'S$ and OPS are similar;

$$\therefore \frac{OP}{OS} = \frac{OS}{OP'}, \quad \therefore OP \,.\, OP' = (2R)^2 \text{ which is a constant.}$$

$\therefore P'$ is the inverse of P in respect to O, and the line $P'S$ is the inverse of the circle $OPSQ$ in respect to O. If the figure is rotated about OS, then P' traces out a tangent plane.

Some important generalisations follow:

(1) *The inverse of a circle which passes through O is a straight line perpendicular to the diameter through O.*

It will be remembered that the stereographic projections of great circles through the view point are straight lines.

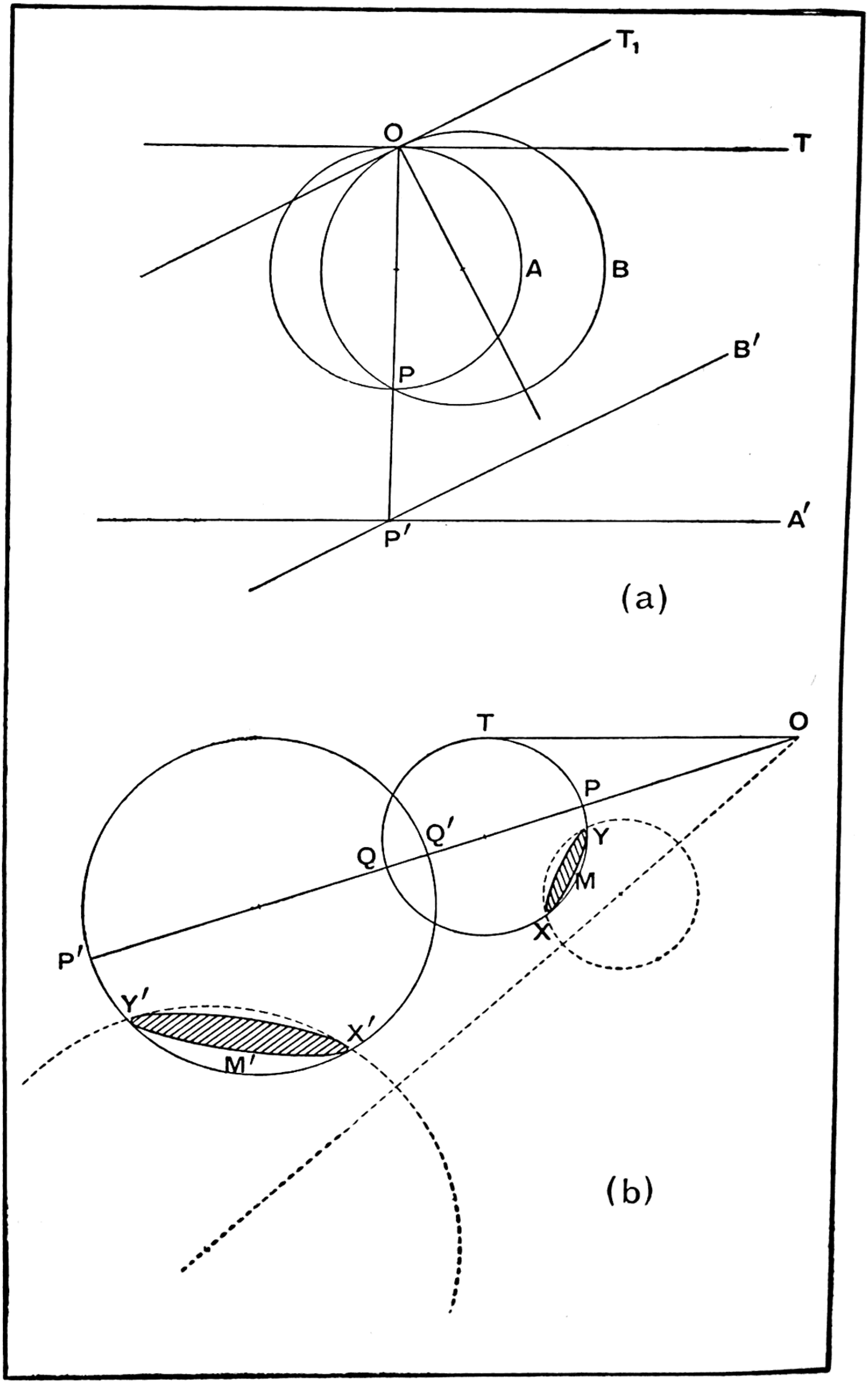


Diagram 51.—Stereographic Projection as Inversion (II)

(2) *Conversely, as P is also the inverse of P', the inverse of a straight line is the circle which passes through O and of which the diameter through O is perpendicular to the line.*

To prove, firstly, that angles in a plane are represented by equal angles in inversion :

Let $P'B'$, $P'A'$ be two straight lines co-planar with O, and the circles OPA, OPB their inverses in respect to O. As diameters of the circles are perpendicular to the lines, they intersect at an angle equal to $A'P'B'$. $\therefore$ Tangents at O intersect at an angle equal to $A'P'B'$. Thus angles are conserved.

Exercise. Consider *any* two co-planar lines. By considering the inversion of this plane and the planes defined by O and each of the lines, prove the general proposition (see p. 114).

Lastly, to show that all circles which do not pass through O invert into circles.

OPQ is a straight line through the centre of the circle PQX, which is the locus of the moving points P and Q.

P' and Q' are the inverses of P and Q in respect to O. It is required to show, firstly, that the locus of P' and Q' is a circle.

$OP \, . \, OQ$ is a constant (OT^2)

$$OP \, . \, OP' = k^2 \, ; \quad OQ \, . \, OQ' = k^2 \, ; \quad \therefore \; OP' \, . \, OQ' = \frac{k^4}{OP \, . \, OQ},$$

which is a constant.

$\therefore$ the locus of P' and Q' is a circle whose centre lies on OP produced.

If the circles rotate about OP', they will trace out the surfaces of two spheres, from which it follows that *the inverse of a sphere, when O is not on its surface, is another sphere.*

Let XMY be any circle on the sphere and $X'M'Y'$ its inverse. As any circle may be regarded as the intersection of two spheres, the figure $X'M'Y'$ will represent the intersection of the two inverse spheres and will therefore be itself a circle.

$\therefore$ *the stereographic projection of any circle which does not pass through the view point is a circle.*

APPENDIX B

TABLES SHOWING DISTANCES FROM THE EQUATOR OF THE PARALLELS IN (*a*) MERCATOR'S AND (*b*) MOLLWEIDE'S PROJECTIONS

Distances are shown for every fifth degree (i) with the radius as unity and (ii) with the radius as 63·66 mm. which is equivalent to a scale of 1/100 million.

(*a*) MERCATOR'S PROJECTION

Lat.	(i)	(ii)	Lat.	(i)	(ii)
5	·0875	5·57 mm.	50	1·0107	64·28 mm.
10	·1757	11·18 mm.	55	1·1542	73·47 mm.
15	·2647	16·85 mm.	60	1·3169	83·83 mm.
20	·3562	22·67 mm.	65	1·5064	95·90 mm.
25	·4509	28·70 mm.	70	1·7259	109·87 mm.
30	·5492	34·96 mm.	75	2·0276	129·08 mm.
35	·6528	41·56 mm.	80	2·4372	155·15 mm.
40	·7629	48·56 mm.	85	3·1312	199·33 mm.
45	·8813	56·10 mm.	90		

(*b*) MOLLWEIDE'S PROJECTION

Lat.	(i)	(ii)	Lat.	(i)	(ii)
5	·068	4·33 mm.	50	·651	41·44 mm.
10	·137	8·72 mm.	55	·708	45·07 mm.
15	·205	13·05 mm.	60	·762	48·51 mm.
20	·272	17·32 mm.	65	·814	51·82 mm.
25	·338	21·52 mm.	70	·862	54·87 mm.
30	·404	25·72 mm.	75	·906	57·68 mm.
35	·468	29·79 mm.	80	·945	60·16 mm.
40	·531	33·80 mm.	85	·978	62·26 mm.
45	·592	37·69 mm.	90	1·000	63·66 mm.

GRADUATED QUESTIONS AND EXERCISES

The sources of questions selected from public examination papers are acknowledged.

1. Define the following terms, in their relation to map projection: *graticule*, *rhumb line*, *azimuthal*, *great circle*, *co-latitude*, *equidistant*, *orthomorphic*, *loxodromic curve*, *normal zenithal*, *standard parallel*.

2. What is the true or mathematical meaning of the term 'projection'? How far is the term, thus defined, correctly applicable to the Simple Conical, Zenithal and Cylindrical systems of map projection?

3. State the essential characteristics of the meridians and parallels as they occur on the globe, and the relations which exist between them.

4. Draw a figure to represent the spherical quadrilateral formed by the intersections of meridians 10° W. and 30° E. with the Equator and parallel of latitude of 60° N. State approximately the length on the globe of each of the four sides and justify your method of calculation.

5. What is the length, on a globe of radius R, of the arc of the parallel of latitude n° which lies between the meridians $m,^\circ$ and $m,,^\circ$? Prove your formula.

6. Discuss the statement that it is possible to regard cylindrical and zenithal projections as types of conical projection.

7. To what extent is it possible to represent shape accurately on a map? How far can the terms *orthomorphic* (correctly-shaped) and *homolographic* (similarly drawn) be literally applied to the class of projections they respectively describe?

8. Explain clearly, with the aid of a diagram, (*a*) why an aeroplane, taking the shortest route between two towns on, say, parallel 50° N., would not fly due east or west, and (*b*) why an aeroplane, taking the shortest route between two towns on, say, meridian 50° E., would fly due north or south.

9. Explain and differentiate between 'straight-line' and 'rhumb-line' navigation. Which projections produce a map respectively suitable for these two objects?

10. What is the nature of the map projection known as Mercator's Projection? Compare this with any *two* other methods, and discuss their relative advantages and disadvantages.

(University of Cambridge, H.S.C.)

11. What conditions have to be fulfilled for a projection to provide an equal-area map-net? Justify your answer.

12. For what special purposes are equal-area maps most useful? State your reasons. What disadvantages have to be tolerated in an equal-area map?

13. What do you understand by (i) an equal-area map-net, (ii) a cylindrical map-net? (*b*) Name one example of each type commonly used in atlases for maps of the world, and in each case state (i) the general arrangement of lines of latitude and longitude, (ii) the chief advantages and disadvantages of the net.

(Northern Universities, H.S.C. Subsidiary.)

14. (*a*) Define cylindrical projection, and enumerate the properties which are common to and characteristic of all cylindrical projections.

(*b*) In what way can a variant be introduced into cylindrical projection so that different species of this class of projections can be derived?

15. What are the characteristics common to all polar zenithal map projections? In what respect do such projections differ?

16. Describe three of the chief types of map projection, and state what type of projection you would choose for a map of the Arctic Regions, giving reasons for your choice.

(University of Cambridge, H.S.C.)

17. What projections would you use for :

(*a*) An air navigation map of Western Europe.
(*b*) A population distribution map of the same region.
(*c*) An atlas map of Africa for general purposes?

(University of Cambridge, H.S.C.)

18. Criticise or justify the use of :

(*a*) A world map on Mercator's Projection for comparing the areas occupied by intertropical forests and forests of the temperate zone.

(*b*) A world map on Mollweide's Projection showing ocean currents and winds.

Which projections are commonly used for the individual countries of Europe, and why?

(Northern Universities, H.S.C.)

19. For what purposes are the following map projections most suitable: Bonne, Mercator, Sanson-Flamsteed, Gnomonic? State briefly your reasons, and give the chief properties of each of these projections.

(Northern Universities, H.S.C.)

20. Construct a table of differences which one could use for the recognition of the most commonly used projections by inspection.

21. Name the projection used (for a given map) and suggest its advantages. Discuss the choice of a suitable projection for a map of Africa and also for a map of the Pacific Ocean.

(University of London, H.S.C.)

22. What projections would you consider suitable for a map of South America? Which would you choose for an atlas map, and why?

23. Discuss the choice of projections available for different kinds of world maps, pointing out in each case advantages and disadvantages of a particular choice.

24. Discuss the utilities and limitations of any equal-area map-nets known to you and describe the construction of any *one* of them.

(Northern Universities, H.S.C. Scholarship Paper.)

25. Compare the construction, appearance and uses of the Polar Zenithal equal-area and the Cylindrical equal-area map networks.

(Northern Universities, H.S.C. Scholarship Paper.)

26. Discuss the essential differences between the Zenithal equidistant and Sinusoidal projections with regard to comparatively narrow areas running approximately equidistant north and south of the Equator. Extend your remarks to the choice between these projections for an airways route map from Egypt to the Cape.

(University of London, B.A. Hons. and B.Sc. Special Examinations for Internal Students.)

27. State the position of the view point in Orthographic, Stereographic, Gnomic and La Hire's Zenithal Projections. Show how to construct graphically any *one* of them.

28. How would you distinguish a normal zenithal projection from any other kind of projection used for a map of an equatorial region? Justify the popularity of Lambert's Zenithal Equal-Area Projection.

29. Explain the method of constructing a zenithal projection by transference from another zenithal projection. On what property or properties of zenithal projection does this method depend?

30. Describe the appearance of a normal Gnomonic Zenithal Projection. In what way or ways does this projection differ from all other normal zenithal projections? Account for this differentiation.

31. Describe, with a diagram, the method of constructing graphically a normal Gnomonic Zenithal Projection. Explain and justify your method.

32. Show how to construct graphically an oblique Stereographic Zenithal Projection. On what general properties of Stereographic Projection does your method depend?

33. What characteristics are common to all oblique zenithal projections other than the Gnomonic. Show, with a diagram, the reasons for the existence of these common properties. What practical use can be made of these common qualities in the actual process of constructing a zenithal projection?

34. What is meant by reference, in zenithal projection, to 'the appropriate trigonometrical scale'? State the scale characteristic of any two zenithal projections and show how, in each case, the scale is derived.

35. Suppose that a trigonometrical scale is being used for the construction, by Lambert's Equal-Area Zenithal Projection, of a map of a portion of the surface of a given globe of radius R. (*a*) State and prove the formula of the appropriate scale; (*b*) if the map is to equal the area of the spherical surface projected, why is it necessary to multiply this scale by $\sqrt{2}R$?

36. Discuss the essential differences between Mollweide's and Sanson-Flamsteed's Projections. How can they be distinguished at sight, even when a relatively small section of each is shown?

37. Justify the use of the term 'sinusoidal' as a description of the Sanson-Flamsteed Projection.

38. Describe the appearance, value and construction of a map of the world on an Interrupted Homolographic Projection based either on Mollweide's or on Sanson-Flamsteed's Projection.

39. Write a concise description of Gall's Projection. Calculate the length of the Equator in a map based on this projection and drawn to a scale of 1 : 100 million ($r = 63{\cdot}66$ mm.).

40. What is meant by the expression 'the constant of the cone' in conical projection? Assume, in a simple conical projection, that the standard parallel is of latitude $n°$; show that the constant of

the cone equals sin $n°$. What will be the length of the arc representing the standard parallel if the radius of the circumscribed sphere is R?

41. Selecting 40° as the standard parallel and a scale of 1 : 200 million ($r = 31{\cdot}8$ mm.), construct the cone and the standard parallel as the basis of a simple conical projection.

42. 'The scale of zenithal equidistant projection, whether polar, normal or oblique, is 1·57 times its true length, or 1·57 times the scale of the projected sphere.' Why is this so?

43. Define the projection known as the Conical with Two Standard Parallels. Show how it may be constructed, selecting the general case in which the selected parallels are of latitude $n_1°$ and $n_2°$ respectively.

44. Calculate the constant of the cone in a conical projection with two standard parallels of latitude 50° N. and 60° N. respectively. (Tables may be used.)

45. Using the value obtained for the constant of the cone (c) in Question 44, calculate the radii of the arcs which are to represent the standard parallels. Assume a scale of 1 : 20 million, when r will equal 318·3 mm.

46. State and derive the formula from which the distances of the parallels from the Equator, in Mollweide's Projection, can be calculated.

47. Given that a parallel of latitude is 16·90 mm. from the Equator in a map on Mollweide's Projection, drawn to a scale of 1 : 200 million ($r = 31{\cdot}83$ mm.), find by calculation the latitude of the parallel.

48. Explain the principle on which a map on Mercator's Projection is constructed. It is not necessary to state the formula which expresses the distances of the parallels from the Equator. Show why the projection is orthomorphic and why it represents compass bearings by straight lines.

49. Describe a simple graphical method of constructing Bonne's Projection and show where the consequent inaccuracy, though slight, exists. Explain why this projection is an equal-area projection.

50. What is the mathematical meaning of Inversion? Show that Stereographic Projection satisfies the conditions implied in the definition and what use can be made of this conception.

ANSWERS TO NUMERICAL QUESTIONS

4. The intersected arc of the Equator is approximately 2767 miles ; that of parallel 60° is approximately 1383 miles ; the arcs of the meridians are approximately 4152 miles.

5. $\dfrac{\pi R(m_{,,}^{\circ} - m_{,}^{\circ})}{180} \cos n^{\circ}$.

39. 283 mm.

44. 0·818° ; i.e. 49′ 5″.

45. 250 mm. and 194·5 mm. respectively.

47. 40°.

Answer to Exercise, page 107.

(1) The inverse of the plane which contains the lines is a sphere through *O*, and its diameter through *O* is perpendicular to the plane.
∴ Tangents to the sphere at *O* are parallel to the plane.

(2) The two planes defined by *O* and each of the lines respectively are unchanged by inversion, and the circles of intersection of the sphere and the planes are inverse to the given lines.

From (1) and (2) it follows that the angle between the circles is equal to the angle between the lines. This is true of any two intersecting lines ;
∴ it is true of any two intersecting curves, as the angle between them at a point of intersection is defined as the angle between their tangents at this point.

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